Electromagnetic open waveguides : mathematical analysis

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Abstract

The study of open electromagnetic waveguides amounts to the spectral analysis of selfadjoint operators with noncompact resolvant. In this article, we are particularly interested in obtaining existence results for guided modes and in studying their properties. The distinguishing point of our work, is that we consider the case where both the dielectric permittivity ε and the magnetic permeability μ may vary in the cross section of the waveguide. The originality of our approach, with respect to previous works (cf. [2]) is that it takes into account the divergence-free condition in the functional framework. In this paper we exhibit various existence results depending on various assumptions on ε and μ, study the properties of the thresholds and point out what happens at low and high frequencies.

Key Words: Electromagnetic Guided Waves, Maxwell's Equations, Spectral Theory, Min-Max Principle, Thresholds, Sobolev Spaces Hrot,Hdiv

Guides d'Ondes Electromagnétiques Ouverts: Analyse Mathématique

Résumé

L'étude des guides électromagnétiques ouverts se ramène à l'analyse spectrale d'une famille d'opérateurs autoadjoints, à résolvantes non compactes. Dans cet article, nous nous intéressons particulièrement à l'obtention de résultats d'exitence de modes guidés ainsi qu' à l'étude de leurs propriétés. La spécificité de notre travail concerne le cas où la permittivité diélectrique ε et la perméabilité magnétique μ varient simultanément dans la section transverse du guide. L'originalité de notre approche, par rapport à des travaux antérieurs, consiste à prendre en compte la condition de divergence nulle dans le cadre fonctionnel. Dans cet article, nous montrons des résultats d'existence suivant les hypothèses vérifiées par ε et μ, étudions les propriétés des fréquences de coupure et plus particulièrement traitons ce qui se passe à basse et haute fréquence.

Mots Clés: Ondes Electromagnétiques Guidées, Equations de Maxwell, Théorie Spectrale, Principe du Min-Max, Fréquences de Coupure, Espaces de Sobolev Hrot,Hdiv

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Introduction

The study of wave propagation phenomena, especially of electromagnetic waves, constitutes a very fertile field of researches in applied mathematics and numerical analysis. In this vast area, the study of waveguides represents an attractive domain from both numerical and theoretical points of view, in particular for the open problems that it raises. The electromagnetic waveguides have applications in various domains of physics (electronic components, optical fibers, integrated optics...) and have been already abundantly studied, for instance by D. Marcuse [22] who is the reference in the physical literature, or by A. Bamberger and A.S. Bonnet [2], A.S. Bonnet [7], R. Djellouli [13], N.Gmati [17], A. Bermúdez and D.G. Pedreira [5], F. Kikuchi [20], concerning the mathematical or numerical studies.

A waveguide is a cylindrical propagation medium infinite in each direction, invariant, with respect to the geometry as well as the physical characteristics of the medium (here, the dielectric permittivity and the magnetic permeability), under any translation in a privileged direction, for instance the x_3 direction.

From the mathematical point of view, the usual objective is the complete spectral theory of a differential selfadjoint operator appearing in the mathematical propagation model. Such a study is in fact a preliminary study for the scattering theory for locally perturbed media. Mathematically the most difficult case is the one of open waveguides. To our knowledge, the complete scattering theory has been carried out completely essentially in the case of stratified media that is to say media which are invariant under any translation not only in one direction but in two space directions. For scalar propagation models let us cite, in order of increasing generality, the works by Y. Dermenjian and J.C. Guillot [11], C.H. Wilcox [32], M. Ben Artzi, Y. Dermenjian and J.C. Guillot [1], S. De Bièvre and D.W. Pravica [6] (this list is far from exhaustive). Concerning electromagnetic waves, we mention J.C. Guillot's work [18],the more complete reference being probably the recent monograph of R. Weder [31]. In this paper we are specifically interested in the study of guided modes in $3D$ electromagnetic media which are invariant under translation in only one space direction. In this case, even the study of the unperturbed media raises some nontrivial questions. More precisely the aim of the present work is to obtain existence results of guided modes and to analyze their properties.

The outline of our article is as follows: guided waves are defined in Section 1. We show that the problem amounts to the spectral analysis of a family of selfadjoint operators. We determine by compact perturbation techniques, their essential spectrum (see Section 2), which is the intermediate phase to study their point spectrum. In Section 3, we show that the essential spectrum does not contain any eigenvalues and characterize the eigenvalues with the help of the Min-Max principle. In Section 4, we apply this characterization in order to obtain a existence result, which leads us to introduce the notion of threshold or cut-off frequency. A lot of results concerning the properties of guided modes are expressed in terms of the thresholds. This is the reason why an in depth study of them is dealt with in Section 5. We are in particular interested to the evolution of numbers of guided modes with respect to β and investigate conditions on the medium for existence or nonexistence of guided modes at low frequencies.

1 Mathematical formulation of the problem

1.1 Position of the problem

We consider a 3D dielectric linear isotropic medium occupying the whole space $\mathbf{R}^3$. We denote by (x, x_3), with $x = (x_1, x_2) \in \mathbf{R}^2$, the generic point of $\mathbf{R}^3$. We assume that the propagation medium has a cylindrical structure in the sense that it is invariant under any translation in the x_3 direction (see figure 1.1). This means that the dielectric permittivity ε and the magnetic permeability μ are functions of the only transverse variable x:

$$\begin{cases} \varepsilon(x, x_3) = \varepsilon(x) \\ \mu(x, x_3) = \mu(x). \end{cases} \tag{1.1}$$

We make the usual assumption on the functions ε and μ: they are measurable, strictly positive and bounded functions. We introduce

$$\begin{cases} \varepsilon_- = \inf_{x\in\mathbf{R}^2} \varepsilon(x) > 0 \quad ; \quad \varepsilon^+ = \sup_{x\in\mathbf{R}^2} \varepsilon(x) < +\infty \\ \mu_- = \inf_{x\in\mathbf{R}^2} \mu(x) > 0 \quad ; \quad \mu^+ = \sup_{x\in\mathbf{R}^2} \mu(x) < +\infty \,. \end{cases} \tag{1.2}$$

Another important property of the propagation medium we shall consider is the fact that each cross section (i.e. parallel to (x_1, x_2)) is homogeneous at infinity. More precisely ε and μ are constant outside some bounded domain B_R (the disc of radius R centered at the origin) of the plane (x_1, x_2):

$$\exists R > 0 \ / \ |x| \geq R \ \Rightarrow \ \varepsilon(x) = \varepsilon_\infty \ , \ \mu(x) = \mu_\infty \,. \tag{1.3}$$

For the sequel it is useful to introduce the local propagation velocity $c(x)$ of the medium, defined by

$$c(x)^2 = (\varepsilon(x)\mu(x))^{-1} \,, \tag{1.4}$$

$c(x)$ is clearly a bounded, strictly positive measurable function and we shall set

$$c_- = \inf_{x\in\mathbf{R}^2} c(x), \quad c_+ = \sup_{x\in\mathbf{R}^2} c(x), \quad c_\infty = (\varepsilon_\infty\mu_\infty)^{-\frac{1}{2}} \,. \tag{1.5}$$

Of course c_∞ is the value of $c(x)$ at infinity: $|x| \geq R \ \Rightarrow \ c(x) = c_\infty$.

The electromagnetic field is as usual described by the electric field $\mathbf{E}(x, x_3, t)$ and the magnetic field $\mathbf{H}(x, x_3, t)$ ($t > 0$ denotes the time) whose variations are governed by Maxwell's equations

$$\begin{cases} \varepsilon \dfrac{\partial \mathbf{E}}{\partial t} - \mathrm{rot}\mathbf{H} = 0 \\ \mu \dfrac{\partial \mathbf{H}}{\partial t} + \mathrm{rot}\mathbf{E} = 0 \,. \end{cases} \tag{1.6}$$

Guided waves are particular solutions of (1.6) on the form

$$\begin{cases} \mathbf{E}(x, x_3, t) = (E_1(x) \ , \ E_2(x) \ , \ -iE_3(x))^T \exp i(\omega t - \beta x_3) \\ \mathbf{H}(x, x_3, t) = (H_1(x) \ , \ H_2(x) \ ; \ iH_3(x))^T \exp i(\omega t - \beta x_3) \end{cases} \tag{1.7}$$

where

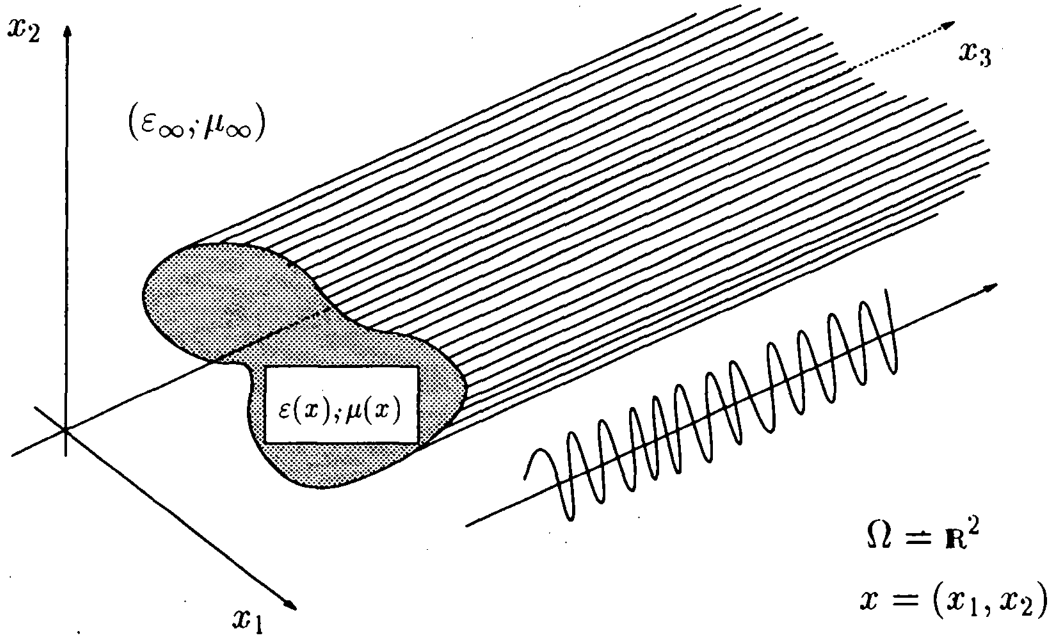

Figure 1.1: Open Waveguide

- $\omega > 0$ is the pulsation of the wave
- $\beta > 0$ is the wavenumber in the x_3-direction

and where the transverse electromagnetic energy is supposed to be finite (we set $E = (E_1, E_2, E_3)^t$ and $H = (H_1, H_2, H_3)^t$)

$$\int_{\mathbf{R}^2} (\varepsilon \mid E \mid^2 + \mu \mid H \mid^2) \, dx < +\infty \, . \tag{1.8}$$

The expression (1.7) represents an harmonic plane wave propagating without any distortion in the direction x_3 with a velocity $V = \omega/\beta$ (the phase velocity). Such a solution is periodic in the direction x_3 and the period $\lambda = 2\pi/\beta$ is called the wavelength. The 2D vector fields (with values in $\mathbf{R}^3$) $E(x)$ and $H(x)$ describe the distribution of the electromagnetic field in each cross section. Guided waves differ from usual plane waves in a homogeneous medium, for instance, by the square integrability condition (1.8) which characterizes the fact that a mode is guided or not. This condition physically means that the energy of the mode remains confined in some bounded region of the cross section: this is where the fact that the coefficients $\varepsilon(x)$ and $\mu(x)$ vary locally plays a fundamental role. Indeed when these coefficients are constant, it is well known that guided waves do not exist. From a theoretical point of view the first fundamental question which naturally arises is the following one:

(i) What conditions on $\varepsilon(x)$ and $\mu(x)$ can ensure the existence of guided waves ?

Another point to emphasize is the fact that guided modes, even when they exist, do not exist for any values of ω and β: ω and β must be linked by some relation $\omega = f(\beta)$ which is called the dispersion relation of the mode (the corresponding curve in the (β,ω)-plane being the dispersion curve). As a consequence the phase velocity $V = \omega/\beta$ is a function of β: guided waves are dispersive. This is the second major difference between usual plane waves in a homogeneous medium (which are not dispersive) and guided waves. Therefore the second natural question is

(ii) What are the properties of the dispersion relation of the guided waves ?

These are the two questions we intend to address in this article, as well as the question of the number of solutions and the related notion of thresholds (or cut-off frequencies) and the problem of asymptotic results at low and high frequency. The only case where one can answer completely to all these questions is the case where the functions ε and μ take two values:

$$\text{(1.9)} \qquad \begin{cases} \varepsilon(x) = \varepsilon_0 \quad \text{for } \mid x \mid < R \quad , \quad \varepsilon_\infty \quad \text{for } \mid x \mid \geq R \\ \mu(x) = \mu_0 \quad \text{for } \mid x \mid < R \quad , \quad \mu_\infty \quad \text{for } \mid x \mid \geq R \end{cases}$$

In this case an analytical solution is available (see for instance J.P. Pocholle [25], D. Marcuse [22]). The theory for the general case is of course much more complicated. Recently, A. Bamberger and A.S. Bonnet [2], with the help of the spectral theory of selfadjoint operators, made a major step in the understanding of the properties of electromagnetic waveguides. In fact they obtained very complete results in the specific case where the function $\mu(x)$ is constant everywhere. This assumption is physically relevant in many applications and has a very important technical consequence: the H^1-regularity of the magnetic field H as a function of x. This is the reason why the authors in [2] considered a formulation of the problem in which H was the unknown: adding an artificial term in the corresponding variational formulation, they got rid of the problem of the divergence-free condition and were able to develop the theory in the H^1 functional framework. In the general case where $\varepsilon(x)$ and $\mu(x)$ vary simultaneously, with possible discontinuities, this is no longer possible. It is precisely one of the purposes of the present work to overcome this difficulty. Towards this goal we shall give a new formulation of the problem in which the divergence-free condition is included in the functional framework. With such a formulation we can work equivalently with the electric field or the magnetic field which allows us to take profit from the natural symmetry of Maxwell's equations with respect to E and H. This will lead us to the generalization of the results of A. Bamberger and A.S. Bonnet to the case $\mu(x)$ variable. The reader will easily check that our results coïncide with those of [2] when $\mu(x) = \mu_\infty$. The second major interest of this new formulation is to be preparatory for the derivation of a numerical method for the computation of the guided modes. The presentation and the analysis of this method will be the subject of a forthcoming paper.

1.2 Mathematical setting

Before entering the rigorous mathematical treatment, we first need to derive the equations of our problem. Plugging formula (1.7) into (1.6) leads to the following system of equations:

$$\text{(1.10)} \qquad \begin{cases} \text{rot}^*_\beta H = \varepsilon\omega E \\ \text{rot}_\beta E = \mu\omega H \end{cases}$$

where the differential operator rot_β is defined by

$$\text{(1.11)} \qquad \mathrm{rot}_\beta u = \begin{pmatrix} \dfrac{\partial u_3}{\partial x_2} - \beta u_2 \\ \beta u_1 - \dfrac{\partial u_3}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} \end{pmatrix}$$

and rot^*_β is the adjoint of rot_β, also given by $\mathrm{rot}^*_\beta = \mathrm{rot}_{-\beta}$.

One can then give two formulations in terms of a symmetric eigenvalue problem by eliminating H or E in (1.10):

(i) The formulation in E:

$$\text{(1.12)} \qquad \varepsilon^{-1}\mathrm{rot}^*_\beta(\mu^{-1}\mathrm{rot}_\beta E) = \omega^2 E$$

(ii) The formulation in H:

$$\text{(1.13)} \qquad \mu^{-1}\mathrm{rot}_\beta(\varepsilon^{-1}\mathrm{rot}^*_\beta H) = \omega^2 H \ ;$$

To complete our presentation, we need a functional framework. Let us consider for instance the E-formulation. We introduce the Hilbert space

$$\text{(1.14)} \qquad H_\varepsilon = L^2(\mathbf{R}^2)^3$$

that we equip with the scalar product

$$\text{(1.15)} \qquad (u,v)_\varepsilon = \int_{\mathbf{R}^2} \varepsilon u.v \, dx.$$

In the sequel, we shall denote for any 3D vector field $u(x) = (u_1(x), u_2(x), u_3(x))$ the transverse field by $\boldsymbol{u} = (u_1, u_2)$ so that we can write indifferently u or $(\boldsymbol{u}, u_3)$. We shall also introduce

$$\text{(1.16)} \qquad V_\varepsilon = \{u \in H_\varepsilon \ / \ \mathrm{rot}_\beta u \in H_\varepsilon\} .$$

It is immediate to verify that

$$\text{(1.17)} \qquad V_\varepsilon = \{u = (\boldsymbol{u}, u_3) \in H(\mathrm{rot}; \mathbf{R}^2) \times H^1(\mathbf{R}^2)\}$$

where as usual (see [15])

$$\begin{cases} H(\mathrm{rot}; \mathbf{R}^2) = \left\{\boldsymbol{u} \in L^2(\mathbf{R}^2)^2 \ / \ \mathrm{rot}\boldsymbol{u} = \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} \in L^2(\mathbf{R}^2)\right\} \\ H^1(\mathbf{R}^2) = \{\varphi \in L^2(\mathbf{R}^2) \ / \ \nabla\varphi \in L^2(\mathbf{R}^2)^2\} . \end{cases}$$

The space V_ε is an Hilbert space for the norm

$$\text{(1.18)} \qquad \| u \|^2_{V_\varepsilon} = \int_{\mathbf{R}^2} (| \boldsymbol{u} |^2 + | \nabla u_3 |^2 + | \mathrm{rot}\boldsymbol{u} |^2) \, dx .$$

Finally we denote by $\tilde{A}_\varepsilon(\beta)$ the unbounded operator in H_ε defined by

$$\text{(1.19)} \qquad \begin{cases} D(\tilde{A}_\varepsilon(\beta)) = \{u \in V_\varepsilon \quad / \quad \mathrm{rot}^*_\beta(\mu^{-1}\mathrm{rot}_\beta u) \in H_\varepsilon\} \\ \tilde{A}_\varepsilon(\beta)u = \varepsilon^{-1}\mathrm{rot}^*_\beta(\mu^{-1}\mathrm{rot}_\beta u), \quad \forall u \in D(\tilde{A}_\varepsilon(\beta)) . \end{cases}$$

Therefore the problem to be solved can be written as, for a given value of the wave number β considered as a parameter

$$(1.20) \qquad \begin{cases} \text{Find } E \in D(\tilde{A}_\varepsilon(\beta)) \text{ and } \omega^2 \in \mathbb{R}^* \quad \text{such that} \\ \tilde{A}_\varepsilon(\beta)E = \omega^2 E, \quad E \neq 0. \end{cases}$$

The problem (1.20) clearly appears as a family of eigenvalue problems parameterized by β in which ω^2 plays the rôle of the eigenvalue and E the rôle of the corresponding eigenvector.

Let us now introduce the differential operator

$$(1.21) \qquad \mathrm{div}_\beta u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} - \beta u_3 = \mathrm{div}\boldsymbol{u} - \beta u_3 \,.$$

One easily checks that

$$(1.22) \qquad \mathrm{div}_\beta(\mathrm{rot}_\beta^*) = 0 \,.$$

Applying (1.22) to the eigenvalue equation (1.12) we obtain that, as soon as $\omega^2 \neq 0$,

$$(1.23) \qquad \mathrm{div}_\beta(\varepsilon E) = 0$$

which means that all physically relevant solutions (i.e. for which $\omega^2 \neq 0$) satisfy the generalized divergence-free condition (1.23). Moreover defining

$$(1.24) \qquad \nabla_\beta \varphi = \left(\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \beta\varphi\right)^t$$

we have

$$(1.25) \qquad \mathrm{rot}_\beta(\nabla_\beta) = 0.$$

This proves that for any φ in $H^1(\mathbb{R}^2)$, $\nabla_\beta\varphi$ belongs to $D(\tilde{A}_\varepsilon(\beta))$ and $\tilde{A}_\varepsilon(\beta)\nabla_\beta\varphi = 0$. To exploit these properties, we shall use the

<u>**Lemma**</u> **1.1** *One has the orthogonal decomposition (with respect to $(.,.)_\varepsilon$)*

$$\forall v \in H_\varepsilon \qquad v = u + \nabla_\beta\varphi \qquad u \in H_\varepsilon(\beta),\ \varphi \in H^1(\mathbb{R}^2)$$

where

$$(1.26) \qquad H_\varepsilon(\beta) \;=\; \{u \in H_\varepsilon \;/\; div_\beta(\varepsilon u) = 0\} \,.$$

Proof Introduce the unique solution φ in $H^1(\mathbb{R}^2)$ of $\mathrm{div}_\beta(\varepsilon\nabla_\beta\varphi) = \mathrm{div}_\beta(\varepsilon v)$ and set $u = v - \nabla_\beta\varphi$. This gives the decomposition. The orthogonality stems up from Green's formula.

The interest of Lemma 1.1 lies in the

<u>**Lemma**</u> **1.2**

$$(i) \qquad Ker\tilde{A}_\varepsilon(\beta) \;=\; \{\nabla_\beta\varphi,\ \varphi \in H^1(\mathbb{R}^2)\}$$

$$(ii) \qquad \mathcal{I}m\tilde{A}_\varepsilon(\beta) \;\subset\; H_\varepsilon(\beta)$$

Proof The inclusion (ii) is a consequence of (1.22). The relation (1.25) implies that $\{\nabla_\beta\varphi, \varphi \in H^1(\mathbf{R}^2)\} \subset Ker\tilde{A}_\varepsilon(\beta)$. Reciprocally, if u belongs to $Ker(\tilde{A}_\varepsilon(\beta))$, then by Green's formula, we have

$$\int_{\mathbf{R}^2} \mu^{-1}|\mathrm{rot}_\beta u|^2\, dx = 0 \quad \Longrightarrow \quad \mathrm{rot}_\beta u = 0\,.$$

Coming back to the definition of rot_β (see (1.11)), we get $u = \nabla_\beta\varphi$, with $\varphi = \frac{1}{\beta}u_3$, which completes the proof.

Combining condition (1.23) and lemma 1.2, it is natural to consider the restriction of the operator $\tilde{A}_\varepsilon(\beta)$ to the space $H_\varepsilon(\beta)$, which is a closed subspace of H_ε (and then an Hilbert space for the inner product $(.,.)_\varepsilon$). We shall consider this restriction as an unbounded operator $A_\varepsilon(\beta)$ in the Hilbert space $H_\varepsilon(\beta)$

$$(1.27)\qquad \begin{cases} D(A_\varepsilon(\beta)) \;=\; \{u \in V_\varepsilon \cap H_\varepsilon(\beta)\ /\ \mathrm{rot}^*_\beta(\mu^{-1}\mathrm{rot}_\beta u) \in L^2(\mathbf{R}^2)^3\} \\ A_\varepsilon(\beta)u \;=\; \varepsilon^{-1}\mathrm{rot}^*_\beta(\mu^{-1}\mathrm{rot}_\beta u)\,. \end{cases}$$

In the sequel we shall need to work with the bilinear form $a_\varepsilon(\beta\,;.,.)$ associated with $A_\varepsilon(\beta)$. This bilinear form is defined on the space

$$(1.28)\qquad V_\varepsilon(\beta) \;=\; V_\varepsilon \cap H_\varepsilon(\beta) = \{(u, u_\varepsilon) \in H(\mathrm{rot}, \mathbf{R}^2) \times H^1(\mathbf{R}^2)\ /\ \mathrm{div}_\beta(\varepsilon u) = 0\}$$

and its expression is given by

$$(1.29)\qquad a_\varepsilon(\beta\,; u, v) = \int_{\mathbf{R}^2} \mu^{-1}\mathrm{rot}_\beta u \cdot \mathrm{rot}_\beta v\; dx \quad \forall (u,v) \in V_\varepsilon(\beta)^2\,.$$

By Green's formula, one has

$$(1.30)\qquad (A_\varepsilon(\beta)u, v)_\varepsilon \;=\; a_\varepsilon(\beta\,; u, v) \quad \forall (u,v) \in D(A_\varepsilon(\beta)) \times V_\varepsilon(\beta).$$

The fact that $a_\varepsilon(\beta\,;.,.)$ is symmetric and positive implies that $A_\varepsilon(\beta)$ is symmetric and positive. We shall see in section 2 that $A_\varepsilon(\beta)$ is selfadjoint and positive definite. Finally, because of (1.23), problem (1.20) can be reduced to

$$(1.31)\qquad \begin{cases} \text{Find } E \in D(A_\varepsilon(\beta)), \omega^2 > 0 \quad / \\ A_\varepsilon(\beta)E \;=\; \omega^2 E \quad , \quad E \neq 0. \end{cases}$$

For each β, we have to determine the point spectrum of $A_\varepsilon(\beta)$. Because of the unboundedness of $\mathbf{R}^2$, the embedding of $D(A_\varepsilon(\beta))$ in H_ε is not compact: this is why the existence of this point spectrum is not a trivial question. Formulation (1.30) is the one we shall use for our analysis. It differs essentially from the one of A. Bamberger and A.S. Bonnet-Ben Dhia [2] in the fact that the generalized divergence-free condition (1.23) has been incorporated in the functional space $H_\varepsilon(\beta)$. This is essential in order to get some local compactness. The other important remark is that although vector fields in $H_\varepsilon(\beta)$ take their values in $\mathbf{R}^3$, $H_\varepsilon(\beta)$ is isomorphic to a space of 2D vector fields. Indeed from the definition (1.21) of the operator div_β, we deduce immediately that the component u_3 is given from the knowledge of the transverse field $\boldsymbol{u} = (u_1, u_2)$:

$$(1.32)\qquad u_3 = \frac{1}{\varepsilon\beta}\mathrm{div}(\varepsilon \boldsymbol{u})$$

so that $H_\varepsilon(\beta)$ is isomorphic to the space

$$\tilde{H}_\varepsilon = \{\boldsymbol{u} \in L^2(\mathbf{R}^2)^2 \;/\; \mathrm{div}(\varepsilon \boldsymbol{u}) \in L^2(\mathbf{R}^2)\} . \tag{1.33}$$

Of course in our presentation we have chosen to privilege the electric field E by eliminating the magnetic field. We could have made the opposite choice and consequently obtained the following dual formulation. First introduce the Hilbert space

$$H_\mu(\beta) = \{u \in L^2(\mathbf{R}^2)^3 \;/\; \mathrm{div}^*_\beta(\mu u) \;=\; 0\} \tag{1.34}$$

equipped with the scalar product

$$(u,v)_\mu = \int_{\mathbf{R}^2} \mu u \cdot v \, dx . \tag{1.35}$$

By definition, the operator div^*_β is given by

$$\mathrm{div}^*_\beta v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \beta v_3 = \mathrm{div} \boldsymbol{v} + \beta v_3 . \tag{1.36}$$

We also introduce the spaces

$$\left\{ \begin{array}{l} V_\mu = \{u = (\boldsymbol{u}, u_3) \in H(\mathrm{rot}, \mathbf{R}^2) \times H^1(\mathbf{R}^2)\} (\equiv V_\varepsilon \,) \\ \\ V_\mu(\beta) = V_\mu \cap H_\mu(\beta) \end{array} \right. \tag{1.37}$$

that we both equip with the norm $\|\;.\;\|_{V_\varepsilon}$ defined in (1.18). Finally we introduce the unbounded operator $A_\mu(\beta)$

$$\left\{ \begin{array}{l} D(A_\mu(\beta)) = \{u \in V_\mu(\beta) \quad / \quad \mathrm{rot}_\beta(\varepsilon^{-1}\mathrm{rot}^*_\beta u) \in L^2(\mathbf{R}^2)^3\} \\ \\ A_\mu(\beta) u \;=\; \mu^{-1}\mathrm{rot}_\beta(\varepsilon^{-1}\mathrm{rot}^*_\beta u) \quad \forall u \in D(A_\mu(\beta)) \end{array} \right. \tag{1.38}$$

which is associated to the bilinear form

$$a_\mu(\beta; u, v) \;=\; \int_{\mathbf{R}^2} \varepsilon^{-1}\mathrm{rot}^*_\beta u \cdot \mathrm{rot}^*_\beta v \, dx . \tag{1.39}$$

Then the H-formulation of our problem can be written:

$$\left\{ \begin{array}{l} \text{Find } H \in D(A_\mu(\beta)), \omega^2 > 0 \quad / \\ \\ A_\mu(\beta) H \;=\; \omega^2 H, \quad H \neq 0 \end{array} \right. \tag{1.40}$$

This is a point spectrum problem for the operator $A_\mu(\beta)$. It is useful to notice that to pass from one of the operators $A_\varepsilon(\beta)$ or $A_\mu(\beta)$ to the other, it suffices to exchange the rôles of ε and μ and to change β into $-\beta$. This is why the spectral theory of $A_\mu(\beta)$ can be easily deduced from the one of $A_\varepsilon(\beta)$. As an illustration, we can emphasize the equivalence between the two point spectrum problems (1.31) and (1.40) by the following theorem.

Theorem 1.1 *The operators $A_\varepsilon(\beta)$ and $A_\mu(\beta)$ have the same nonzero eigenvalues. More precisely,* *(i) if E is an eigenfunction of $A_\varepsilon(\beta)$ associated with the eigenvalue $\omega^2 > 0$, then $H = \mu^{-1}\mathrm{rot}_\beta E$ is an eigenfunction of $A_\mu(\beta)$ with the same eigenvalue.*

(ii) If H is an eigenfunction of $A_\mu(\beta)$ associated to the eigenvalue $\omega^2 > 0$, then $E = \varepsilon^{-1}\mathrm{rot}^*_\beta H$ is an eigenfunction of $A_\varepsilon(\beta)$ with the same eigenvalue.

(iii) Moreover the eigenspaces of $A_\varepsilon(\beta)$ and $A_\mu(\beta)$ associated to the eigenvalue $\omega^2 > 0$ have the same dimension.

Proof We shall prove only (i) and (iii). The proof of (ii) is completely symmetric. First note that if E is an eigenvalue of $A_\varepsilon(\beta)$ associated to $\omega^2 > 0$ then $\mathrm{rot}_\beta E \neq 0$. Indeed one has the equality

$$\int_{\mathbf{R}^2} \mu^{-1} \mid \mathrm{rot}_\beta E \mid^2 \, dx = \omega^2 \int_{\mathbf{R}^2} \varepsilon \mid E \mid^2 \, dx \, .$$

Let us set $H = \mu^{-1}\mathrm{rot}_\beta E \in L^2(\mathbf{R}^2)^3$. The equality

$$\varepsilon^{-1}\mathrm{rot}^*_\beta H = \omega^2 E \tag{1.41}$$

proves that H belongs to V_μ. Moreover as $\mathrm{div}^*_\beta(\mathrm{rot}_\beta) = 0$ we see that $\mathrm{div}^*_\beta(\mu H) = 0$ which shows that H belongs to $V_\mu(\beta)$. Finally, applying the operator $\mu^{-1}\mathrm{rot}_\beta$ to (1.41) we obtain

$$\mu^{-1}\mathrm{rot}_\beta(\varepsilon^{-1}\mathrm{rot}^*_\beta H) = \omega^2 H$$

which proves that $H \in D(A_\mu(\beta))$ and that $A_\mu(\beta)H = \omega^2 H$. To prove (iii), it suffices to show that two linearly independent eigenvectors e_1, e_2 of $A_\varepsilon(\beta)$ associated to ω^2 generate two linearly independent eigenvectors $h_1 = \mu^{-1}\mathrm{rot}_\beta e_1$ and $h_2 = \mu^{-1}\mathrm{rot}_\beta e_2$ of $A_\mu(\beta)$ associated to ω^2. Indeed we deduce the result from the fact that $\mu^{-1}\mathrm{rot}_\beta e = 0$ and $e \in H_\varepsilon(\beta)$ imply $e = 0$ (see Lemma 1.2).

In the sequel we shall be led to play with the duality between $A_\varepsilon(\beta)$ and $A_\mu(\beta)$. This is why we shall state most of our theorems for both operators even after having proven them only for $A_\varepsilon(\beta)$.

1.3 Additional notations and useful results

(i) In 2 dimensions there exist two rotationals. The scalar one is applied to a vector valued field $\boldsymbol{u}$ and is denoted by $\mathrm{rot}\boldsymbol{u} = \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2}$. The vector valued one is applied to a scalar function ψ and denoted by $\vec{\mathrm{rot}}\psi = (\dfrac{\partial \psi}{\partial x_2}, -\dfrac{\partial \psi}{\partial x_1})$. We can relate these two operators by the duality property:

$$\forall(\boldsymbol{u}, \psi) \in \mathcal{D}(\mathbf{R}^2)^2 \times \mathcal{D}(\mathbf{R}^2) \quad \int_{\mathbf{R}^2} \boldsymbol{u} \cdot \vec{\mathrm{rot}}\psi \, dx = \int_{\mathbf{R}^2} \mathrm{rot}\boldsymbol{u} \, \psi \, dx \, .$$

It is well known that $\mathrm{rot}(\nabla\psi)$ is equal to 0. There exists a reciprocal to this result known as the Poincaré's lemma (see R. Dautray et J.L. Lions chapter IX A. p263 [10], V. Girault et P.A. Raviart [15] for instance):

$$\boldsymbol{u} \in L^2(\Omega)^2 \quad / \quad \mathrm{rot}\boldsymbol{u} = 0 \quad \Longrightarrow \quad \exists\psi \in H^1(\Omega) \quad / \quad \boldsymbol{u} = \nabla\psi$$

where Ω is a simply connected and bounded open set Ω.

(ii) In the sequel we shall use unique continuation theorem for Maxwell's equations.For this we shall use the unique continuation theorem for the Laplace operator that can be found for instance in [19], [21]. The precise result is the following:

Unique continuation theorem
Let Ω be a connected open set of $\mathbf{R}^2$ and B some ball included in Ω. Let $u \in H^2_{loc}(\Omega)^i$ (i=2 or 3) such that

$$\begin{cases} |\Delta u| \leq C(|\nabla u| + |u|) \quad a.e. \text{ in } \Omega \\ u \equiv 0 \text{ in } B \end{cases}$$

then u vanishes identically in Ω.

A consequence of this result is the following theorem.

<u>**Theorem**</u> **1.2** *Assume that ε is piecewise Lipschitz continuous and that $\boldsymbol{u} \in L^2_{loc}(\mathbf{R}^2)^2$ satisfies*

$$\begin{cases} div(\varepsilon \boldsymbol{u}) = 0 \\ rot\boldsymbol{u} = 0 \\ \boldsymbol{u} \equiv 0 \textit{ in some open ball } B \end{cases}$$

then $\boldsymbol{u}$ vanishes identically in Ω.

Proof We shall give the proof only when $\varepsilon \in W^{1,\infty}(\mathbf{R}^2)$. The general case is obtained by repeating the argument for each connected component where ε is regular. By Poincaré's Lemma rot$\boldsymbol{u} = 0$ implies $\boldsymbol{u} = \nabla\varphi$ with $\varphi \in H^1_{loc}(\mathbf{R}^2)$ and div$(\varepsilon\boldsymbol{u}) = 0$ implies div$(\varepsilon\nabla\varphi) = 0$. Therefore we have

$$-\Delta\varphi = \frac{\nabla\varepsilon}{\varepsilon} \cdot \boldsymbol{u}.$$

The result then follows by unique continuation.

For other applications of this result we shall be led to make additional regularity assumptions on the coefficients ε and μ, in fact piecewise regularity assumptions as for Lemma 3.1, that we shall refer as assumption (PR):

$$(PR) \quad \begin{cases} \mathbf{R}^2 = \overline{\Omega_0} \cup \overline{\Omega_1} \cup \overline{\Omega_2} \cup \ldots \cup \overline{\Omega_N} \\ \Omega_j \cap \Omega_l = \emptyset \text{ for } j \neq l, \quad \Omega_0 = \{x/|x| > R\} \\ \forall\, 0 \leq j \leq N \quad \exists(\varepsilon_j, \mu_j) \in W^{2,\infty}(\mathbf{R}^2) \times W^{1,\infty}(\mathbf{R}^2) \cup W^{1,\infty}(\mathbf{R}^2) \times W^{2,\infty}(\mathbf{R}^2) \;/ \\ \qquad \varepsilon = \varepsilon_j \text{ in } \Omega_j, \quad \mu = \mu_j \text{ in } \Omega_j\,. \end{cases}$$

Remark 1.1 • *In fact more general unique continuation results for Maxwell's equations have been recently obtained by V. Vogelsang in [29]. The assumptions on ε and μ are slightly weaker than those we shall consider here but rather complicated to describe. That is why we have chosen to restrict ourselves to (PR) for which we are sure of our results.*

- *Even for applying the unique continuation result for the Laplace operator, assumption (PR) is not optimal and could be weakened, provided that we would introduce additional technical conditions on ε and μ. However this assumption is reasonable with respect to practical applications and thus sufficient for our purpose.*

(iii) We shall also use compactness results in spaces of vector fields. Classical versions of such theorems can be found for instance in [8] and [15]. The more sophisticated version we shall use in this paper is due to Ch. Weber [30].

Let Ω be a bounded open set of $\mathbf{R}^2$ satisfying the restricted cone property. Let $\boldsymbol{u}^n$ be a sequence of $L^2(\Omega)^2$ satisfying

$$\left| \begin{array}{lll} \| \boldsymbol{u}^n \| \leq C, & \| \operatorname{rot} \boldsymbol{u}^n \| \leq C, & \| \operatorname{div}(\varepsilon \boldsymbol{u}^n) \| \leq C \\ \boldsymbol{u}^n \wedge \nu \,|_{\partial\Omega} = 0 & \text{or} & \boldsymbol{u}^n \cdot \nu \,|_{\partial\Omega} = 0 \end{array} \right.$$

(ν denotes here the normal unit vector to $\partial\Omega$) then there exists a subsequence, still denoted $\boldsymbol{u}^n$, such that

$$\boldsymbol{u}^n \longrightarrow \boldsymbol{u} \quad \text{in } L^2(\Omega)^2.$$

2 Selfadjointness - Essential spectrum

For proving the selfadjointness of $A_\varepsilon(\beta)$, we shall use the following

Lemma 2.1 *The bilinear form $a_\varepsilon(\beta;.,.)$ is coercive in the space $V_\varepsilon(\beta)$. More precisely we have*

$$\text{(2.1)} \quad a_\varepsilon(\beta;u,u) \geq c_-^2 \int_{\mathbf{R}^2} \varepsilon(|\, \operatorname{rot} \boldsymbol{u} \,|^2 + |\, \nabla u_3 \,|^2)\, dx + c_-^2 \beta^2 \int_{\mathbf{R}^2} \varepsilon(|\, \boldsymbol{u} \,|^2 + 2 \,|\, u_3 \,|^2)\, dx\ .$$

Proof By definition we have

$$\left| \begin{array}{rcl} a_\varepsilon(\beta;u,u) & = & \displaystyle\int_{\mathbf{R}^2} \mu^{-1} \,|\, \operatorname{rot}_\beta u \,|^2 \; dx = \int_{\mathbf{R}^2} \varepsilon c^2 \,|\, \operatorname{rot}_\beta u \,|^2 \; dx \\ & \geq & \displaystyle c_-^2 \int_{\mathbf{R}^2} \varepsilon \,|\, \operatorname{rot}_\beta u \,|^2 \; dx \\ & = & \displaystyle c_-^2 \int_{\mathbf{R}^2} \varepsilon(|\, \nabla u_3 - \beta \boldsymbol{u} \,|^2 + |\, \operatorname{rot} \boldsymbol{u} \,|^2)\, dx \\ & = & \displaystyle c_-^2 \left\{ \int_{\mathbf{R}^2} \varepsilon(|\, \nabla u_3 \,|^2 + |\, \operatorname{rot} \boldsymbol{u} \,|^2 + \beta^2 \,|\, \boldsymbol{u} \,|^2)\, dx - 2\beta \int_{\mathbf{R}^2} \varepsilon \boldsymbol{u}.\nabla u_3 \, dx \right\}\ . \end{array} \right.$$

By Green's formula, since $\operatorname{div}(\varepsilon \boldsymbol{u}) = \varepsilon \beta u_3$, we have

$$-\int_{\mathbf{R}^2} \varepsilon \boldsymbol{u} \cdot \nabla u_3 \, dx = \int_{\mathbf{R}^2} \operatorname{div}(\varepsilon \boldsymbol{u}) u_3 \, dx = \beta \int_{\mathbf{R}^2} \varepsilon \,|\, u_3 \,|^2 \; dx$$

from which the result follows immediately.

From classical characterizations of selfadjoint operators we deduce

Theorem 2.1 *For any $\beta > 0$, the operator $A_\varepsilon(\beta)$ is selfadjoint, bounded from below. Moreover if $\sigma(A_\varepsilon(\beta))$ denotes the spectrum of $A_\varepsilon(\beta)$, we have the inclusion*

$$\sigma(A_\varepsilon(\beta)) \subset [c_-^2\beta^2, +\infty)$$

Remark 2.1 *In fact the inequality (2.1) also proves that $c_-^2\beta^2$ cannot be an eigenvalue of $A_\varepsilon(\beta)$. Indeed $a_\varepsilon(\beta\ ; u, u) - c_-^2\beta^2 \mid u \mid_\varepsilon^2 \geq \int_{\mathbf{R}^2} \varepsilon c_-^2(\mid rot\boldsymbol{u} \mid^2 + \mid \nabla u_3 \mid^2 + \beta^2 \mid u_3 \mid^2)\, dx$. If u is an eigenfunction of $A_\varepsilon(\beta)$ associated to the eigenvalue $c_-^2\beta^2$, we see that $u_3 = 0$ and $rot\boldsymbol{u} = 0$. As $div(\varepsilon\boldsymbol{u}) = \varepsilon\beta u_3$ we deduce that $div(\varepsilon\boldsymbol{u}) = 0$ and thus that $\boldsymbol{u} \in \mathbf{P}_\varepsilon \cap L^2(\Omega)^2$, where the space $\mathbf{P}_\varepsilon$ is defined in (5.29). This implies that $\boldsymbol{u} = 0$, as we will see in remark 5.4.*

Corollary 2.1 *For any $\beta > 0$, the operator $A_\mu(\beta)$ is selfadjoint, bounded from below and its spectrum $\sigma(A_\mu(\beta))$ satisfies*

$$\sigma(A_\mu(\beta)) \subset [c_-^2\beta^2, +\infty)\ .$$

Moreover $c_-^2\beta^2$ cannot be an eigenvalue of $A_\mu(\beta)$.

We are now going to determine the essential spectrum of $A_\varepsilon(\beta)$. When $\varepsilon = \varepsilon_\infty$ and $\mu = \mu_\infty$, i.e. in the case of a homogeneous medium, the corresponding operator we shall denote by $A_\infty(\beta)$ has a purely continuous spectrum that one determines very easily using Fourier transform. One gets $\sigma(A_\infty(\beta)) = [c_\infty^2\beta^2, +\infty)$. The idea is to prove that $\sigma_{ess}(A_\varepsilon(\beta)) = \sigma(A_\infty(\beta))$ using compact perturbation techniques. We first prove the

Lemma 2.2

$$\sigma_{ess}(A_\varepsilon(\beta)) \supset [c_\infty^2\beta^2, +\infty)\ .$$

Proof We are going to prove that any real number λ greater than $c_\infty^2\beta^2$ belongs to $\sigma(A_\infty(\beta))$ by constructing explicitly an associated singular sequence. The idea of the construction starts from plane waves propagating in a homogeneous medium that we truncate and shift appropriately in order to avoid the perturbation (i.e. the region where the coefficients are not constant). Note that we cannot directly refer to the proof of [2] since we have to take into account the generalized divergence-free condition. Let $k = (k_1, k_2) \in \mathbf{R}^2$ and set

$$u_k(x) = (-k_2\cos(k.x), k_1\cos(k.x), 0)$$

for which one easily verifies that

$$\begin{cases} \mathrm{div}_\beta u_k = 0 \\ \mathrm{rot}_\beta^*(\mathrm{rot}_\beta u_k) = (\beta^2 + k^2)u_k\ . \end{cases}$$

Let us choose k such that $\lambda = c_\infty^2(\beta^2 + |k|^2)$. Now let ψ be a cut-off function satisfying

$$\begin{cases} \psi \in C_0^\infty(\mathbf{R}^2), \quad 0 \leq \psi \leq 1 \\ \mathrm{supp}\ \psi \subset \{x \in \mathbb{R}^2 / R \leq \mid x \mid \leq 4R\} \\ \psi(x) \equiv 1 \quad \text{for } 2R \leq \mid x \mid \leq 3R\ . \end{cases}$$

We construct the sequence $u^n = (\boldsymbol{u}^n, u_3^n)$ such that $\mathrm{div}_\beta(\varepsilon u^n) = 0$, by taking

$$\begin{cases} \boldsymbol{u}^n(x) &= C^n \psi(\frac{x}{n}) \boldsymbol{u}_k(x) \\ u_3^n(x) &= \frac{C^n}{n\beta} \nabla\psi(\frac{x}{n}) \cdot \boldsymbol{u}_k(x) . \end{cases}$$

As ε and μ are constant outside $|x| < R$, it is clear that the sequence u^n belongs to $D(A_\varepsilon(\beta))$ for $n \geq 1$. We choose the normalization constant $C^n > 0$ such as

$$|u^n|_\varepsilon^2 = \int_{\mathbf{R}^2} \varepsilon |u^n|^2 \, dx = 1 .$$

A simple calculation shows that

$$C^n \sim \frac{C}{n}, \quad C > 0 .$$

It follows that $\| u^n \|_{L^\infty} \searrow 0$ and thus ,as $|u^n|_\varepsilon = 1$, that u^n converges to 0 weakly in H_ε. To conclude that $c_\infty^2(\beta^2 + k^2)$ belongs to the essential spectrum of $A_\varepsilon(\beta)$ we only have to prove that

$$|A_\varepsilon(\beta)u^n - c_\infty^2(\beta^2 + k^2)u^n|_\varepsilon \longrightarrow 0.$$

As supp $u^n \subset \{| x | \geq R\}$ and as $\mathrm{div}_\beta u^n = 0$, we deduce that $A_\varepsilon(\beta)u^n = c_\infty^2(-\Delta u^n + \beta^2 u^n)$. One easily checks that $A_\varepsilon(\beta)u^n - \lambda u^n$ may be written as follows:

$$\begin{cases} -c_\infty^2(\Delta \boldsymbol{u}^n + |k|^2 \boldsymbol{u}^n) &= -c_\infty^2 C^n \left(\frac{1}{n^2} \Delta\psi(\frac{x}{n}) u_k(x) + \frac{2}{n} \nabla\psi(\frac{x}{n}) \cdot \nabla \boldsymbol{u}_k(x) \right) \\ -c_\infty^2(\Delta u_3^n + |k|^2 u^n) &= -\frac{c_\infty^2 C^n}{\beta} \left(\frac{1}{n^3} \nabla(\Delta\psi)(\frac{x}{n}) \cdot \boldsymbol{u}_k(x) + \frac{2}{n^2} \sum_{i=1}^{2} \nabla \frac{\partial\psi}{\partial x_i}(\frac{x}{n}) \cdot \frac{\partial \boldsymbol{u}_k}{\partial x_i}(x) \right) \end{cases}$$

from which it is easy to show by explicit computation that

$$| A_\varepsilon(\beta)u^n - c_\infty^2(\beta^2 + k^2)u^n |_\varepsilon \leq \frac{C}{n}$$

(The main reason is the presence of the factors in $\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}$ occurring from the differentiation of the cut-off function $\psi(\frac{x}{n})$). This completes the proof.

For proving the reverse inclusion we shall use a very important decomposition of the quadratic form $a_\varepsilon(\beta \, ; u, u)$.

Proposition 2.1 *One has*

(2.2) $\qquad \forall u \in V_\varepsilon(\beta) \quad a_\varepsilon(\beta \, ; u, u) = c_\infty^2 \beta^2 \mid u \mid_\varepsilon^2 + p_\varepsilon(\beta \, ; u, u) + c_\varepsilon(\beta \, ; u, u)$

where we have defined

$$(2.3) \quad \begin{cases} p_\varepsilon(\beta \, ; u, u) &= \int_{\mathbf{R}^2} \varepsilon(c^2 \mid \mathbf{rot}\, \boldsymbol{u} \mid^2 + c^2 \mid \nabla u_3 \mid^2 + \beta^2 c_\infty^2 \mid u_3 \mid^2) \, dx \\ c_\varepsilon(\beta \, ; u, u) &= \beta^2 \int_{\mathbf{R}^2} \varepsilon(c^2 - c_\infty^2) \mid \boldsymbol{u} \mid^2 \, dx - 2\beta \int_{\mathbf{R}^2} \varepsilon(c^2 - c_\infty^2) \nabla u_3 \cdot \boldsymbol{u} \, dx \end{cases}$$

which have the following properties:

$$\left| \begin{array}{ll} (i) & p_\varepsilon(\beta\ ; u,u) \geq 0 \quad \forall u \in V_\varepsilon(\beta) \\ (ii) & u^n \rightharpoonup u \text{ in } V_\varepsilon(\beta) \text{ weakly } \Longrightarrow \lim\limits_{n\to+\infty} c_\varepsilon(\beta\ ; u^n,u^n) = c_\varepsilon(\beta\ ; u,u) \\ & \text{(up to the extraction of a subsequence).} \end{array} \right.$$

Proof We start from the identity (see the proof of Lemma 2.1)

$$(2.4) \quad \left| \begin{array}{l} a_\varepsilon(\beta\ ; u,u) = \displaystyle\int_{\mathbf{R}^2} \varepsilon c^2 \{| \operatorname{rot} \boldsymbol{u}\ |^2 + |\ \nabla u_3\ |^2 + \beta^2\ |\ \boldsymbol{u}\ |^2\}\ dx \\ \qquad -2\beta \displaystyle\int_{\mathbf{R}^2} \varepsilon c^2 \nabla u_3 \cdot \boldsymbol{u}\ dx\ . \end{array} \right.$$

We transform the last term of (2.4) as follows:

$$(2.5) \quad \left| \begin{array}{rcl} -\displaystyle\int_{\mathbf{R}^2} \varepsilon c^2 \nabla u_3 . \boldsymbol{u}\ dx & = & -\displaystyle\int_{\mathbf{R}^2} \varepsilon (c^2 - c_\infty^2) \nabla u_3 \cdot \boldsymbol{u}\ dx - c_\infty^2 \int_{\mathbf{R}^2} \varepsilon \nabla u_3 . \boldsymbol{u}\ dx \\ & = & -\displaystyle\int_{\mathbf{R}^2} \varepsilon (c^2 - c_\infty^2) \nabla u_3 \cdot \boldsymbol{u}\ dx + \beta c_\infty^2 \int_{\mathbf{R}^2} \varepsilon\ |\ u_3\ |^2\ dx \end{array} \right.$$

(We have used integration by parts and the fact that $\operatorname{div}(\varepsilon \boldsymbol{u}) = \beta u_3$). Therefore as $\displaystyle\int_{\mathbf{R}^2} \varepsilon c^2\ |\ u\ |^2\ dx$ can be split into the sum of $c_\infty^2 \displaystyle\int_{\mathbf{R}^2} \varepsilon |u|^2\ dx$ and $\displaystyle\int_{\mathbf{R}^2} \varepsilon (c^2 - c_\infty^2) |u|^2\ dx$, the decomposition (2.2) follows from (2.4) and (2.5). As property (i) is immediate we only have to check property (ii). Assume that $u^n \rightharpoonup u$ weakly in $V_\beta(\varepsilon)$, this means in particular that

$$\left\{ \begin{array}{l} u_3^n \text{ is bounded in } H^1(\mathbf{R}^2) \\ \boldsymbol{u}^n \text{ is bounded in } H(\operatorname{rot}; \mathbf{R}^2) \\ \operatorname{div}(\varepsilon \boldsymbol{u}^n) \text{ is bounded in } L^2(\mathbf{R}^2) \end{array} \right.$$

Let $B_R = \{x /\ |\ x\ | \leq R\}$, as the embedding from $H^1(\mathbf{R}^2)$ into $L^2(B_R)$ is compact, we can extract from u_3^n a subsequence, still denoted by u_3^n, such as

$$(2.6) \qquad u_3^n \to u_3 \text{ strongly in } L^2(B_R)\ .$$

For $\boldsymbol{u}^n$, we need another compactness result we shall state in a proposition since we shall reuse it in the sequel.

<u>Proposition</u> 2.2 *Let us introduce the Hilbert space*

$$(2.7) \qquad H(rot, div_\varepsilon, \mathbf{R}^2) = \{\boldsymbol{u} \in L^2(\mathbf{R}^2)^2\ /\ rot\boldsymbol{u} \in L^2(\mathbf{R}^2)\ ,\ div(\varepsilon \boldsymbol{u}) \in L^2(\mathbf{R}^2)\}$$

equipped with the norm

$$(2.8) \qquad \|\ \boldsymbol{u}\ \|_\varepsilon^2 = \int_{\mathbf{R}^2} (|\ \boldsymbol{u}\ |^2 + |\ rot\boldsymbol{u}\ |^2 + |\ div(\varepsilon \boldsymbol{u})\ |^2)\ dx.$$

Then the mapping $\boldsymbol{u} \to \boldsymbol{u}\ |_{B_R}$ is compact from $H(rot, div_\varepsilon, \mathbf{R}^2)$ into $L^2(B_R)^2$.

Let us admit for a while this result. We can then assume that the subsequence u^n is such that

(2.9) $$u^n \longrightarrow u \text{ strongly in } L^2(B_R)^2 .$$

Therefore, as $(c^2 - c_\infty^2)$ has compact support included in B_R it follows that

$$\left| \begin{array}{l} \lim\limits_{n\to+\infty} \displaystyle\int_{\mathbf{R}^2} \varepsilon(c^2 - c_\infty^2) \mid u^n \mid^2 \, dx = \int_{\mathbf{R}^2} \varepsilon(c^2 - c_\infty^2) \mid u \mid^2 \, dx \\ \lim\limits_{n\to+\infty} \displaystyle\int_{\mathbf{R}^2} \varepsilon(c^2 - c_\infty^2) \nabla u_3^n . u^n \, dx = \int_{\mathbf{R}^2} \varepsilon(c^2 - c_\infty^2) \nabla u_3 \cdot u \, dx \end{array} \right.$$

which completes the proof of proposition 2.1.

We now give the

Proof of proposition 2.2 Let u^n be a bounded sequence in $H(\text{rot}, \text{div}_\varepsilon, \mathbf{R}^2)$. Let $R' > R$ and ϕ be a cut-off function satisfying

$$\begin{cases} \phi \in C_0^\infty(\mathbf{R}^2), & 0 \le \phi \le 1 \\ \phi(x) = 1 \text{ in } B_R, & \text{supp } \phi \subset B_{R'}. \end{cases}$$

Let $v^n = \phi u^n$, we have

$$\begin{cases} \displaystyle\int_{B_{R'}} |v^n|^2 \, dx \le C, \quad \int_{B_{R'}} |\text{rot} v^n|^2 \, dx \le C, \quad \int_{B_{R'}} |\text{div}(\varepsilon v^n)|^2 \, dx \le C \\ v^n \wedge n \,|_{\partial B_{R'}} = 0 \end{cases}$$

(n denotes here the unit normal vector to $\partial B_{R'}$ and the bound C depends on ϕ). We can then use the result of Ch. Weber [30] (cf section 1.3 (iii)), to assert that one can extract from v^n a subsequence, still denoted by v^n, such that

$$v^n \to v \quad \text{in } L^2(B_{R'}) \text{ strongly}$$

Then defining $u = v\,|_{B_R}$, we have

$$u^n \to u \text{ in } L^2(B_R) \text{ strongly}$$

which completes the proof of proposition 2.2.

Remark 2.2 *To apply the compactness result of Ch. Weber, it is fundamental to have some control on the tangential trace of the vector field on the boundary. This is why we have used a truncation procedure on some open set larger than B_R (here $B_{R'}$).*

We can now prove

Lemma 2.3

$$\inf \sigma_{ess}(A_\varepsilon(\beta)) \ge c_\infty^2 \beta^2$$

Proof Once, one has the decomposition (2.2), the positivity property (i) and compactness result (ii). The proof of Lemma 2.3 follows an approach which is now rather standard (see [4],[3],[12]). We include it here for completeness. Let $\sigma \in \sigma_{ess}(A_\varepsilon(\beta))$, there exists a sequence u^n in $D(A_\varepsilon(\beta))$ such that

$$(2.10) \qquad \begin{cases} u^n \rightharpoonup 0 & \text{weakly in } H_\varepsilon(\beta) \\ A_\varepsilon(\beta)u^n - \sigma u^n \to 0 & \text{strongly in } H_\varepsilon(\beta) \\ \mid u^n \mid_\varepsilon = 1 . \end{cases}$$

Therefore $\lim_{n\to+\infty} a_\varepsilon(\beta\ ; u^n, u^n) = \sigma$ and from the coercivity result of Lemma 2.1, we deduce that u^n is bounded in $V_\varepsilon(\beta)$ so that we can assume that u^n converges weakly to 0 in $V_\varepsilon(\beta)$. By (2.2), we have

$$\left| \begin{array}{lll} a_\varepsilon(\beta\ ; u^n, u^n) & = & c_\infty^2\beta^2 + p_\varepsilon(\beta\ ; u^n, u^n) + c_\varepsilon(\beta\ ; u^n, u^n) \\ & \geq & c_\infty^2\beta^2 + c_\varepsilon(\beta\ ; u^n, u^n) . \end{array} \right.$$

Taking the limit of this inequality when $n \to +\infty$, we get as $\lim_{n\to+\infty} c_\varepsilon(\beta\ ; u^n, u^n) = 0$ (by property (ii))

$$\sigma \geq c_\infty^2\beta^2 \qquad \forall\sigma \in \sigma_{ess}(A(\beta))$$

which yields the result of Lemma (2.3).

Regrouping Lemmas 2.2 and 2.3 and playing with the duality between $A_\varepsilon(\beta)$ and $A_\mu(\beta)$, we can demonstrate the following theorem (Note that $c_\infty^2\beta^2$ does not change if we permute ε and μ):

Theorem 2.2

$$\sigma_{ess}(A_\varepsilon(\beta)) = \sigma_{ess}(A_\mu(\beta)) = [c_\infty^2\beta^2, +\infty)$$

An immediate consequence of Theorem (2.2) is

Corollary 2.2 *The operators $A_\varepsilon(\beta)$ and $A_\mu(\beta)$ have the same spectrum.*

Proof Elements of the spectrum which are not in the essential spectrum are necessarily strictly positive eigenvalues (for $A_\varepsilon(\beta)$ as well as for $A_\mu(\beta)$). From Theorem (2.2), $A_\varepsilon(\beta)$ and $A_\mu(\beta)$ have the same essential spectrum. As by Theorem 1.1 we know that they have the same eigenvalues, one concludes immediately.

3 Characterization of the point spectrum

From the general theory of selfadjoint operators we know that eigenvalues of $A_\varepsilon(\beta)$ are either isolated eigenvalues of finite multiplicity, i.e. elements of the discrete spectrum $\sigma_d(A_\varepsilon(\beta))$, or eigenvalues embedded in the essential spectrum. In fact the latter cannot exist, except possibly for the lower bound $c_\infty^2\beta^2$.

Lemma 3.1 *Under the regularity assumption (PR) (see section 1.3), the operators $A_\varepsilon(\beta)$ and $A_\mu(\beta)$ have no eigenvalues in the interval $(c_\infty^2\beta^2, +\infty)$.*

Proof Let us first recall what is the regularity assumption (PR):

$$(PR)\quad \begin{cases} \mathbf{R}^2 = \overline{\Omega_0} \cup \overline{\Omega_1} \cup \overline{\Omega_2} \cup \ldots \cup \overline{\Omega_N} \\ \Omega_j \cap \Omega_l = \emptyset \text{ for } j \neq l, \quad \Omega_0 = \{x/|x| > R\} \\ \forall\, 0 \leq j \leq N \quad \exists (\varepsilon_j, \mu_j) \in W^{2,\infty}(\mathbf{R}^2) \times W^{1,\infty}(\mathbf{R}^2) \cup W^{1,\infty}(\mathbf{R}^2) \times W^{2,\infty}(\mathbf{R}^2) / \\ \qquad \varepsilon = \varepsilon_j \quad \text{in } \Omega_j, \mu = \mu_j \quad \text{in } \Omega_j\ . \end{cases}$$

We shall assume, which is not réstrictive that the open sets Ω_j are connected and numbered in such a way that the open sets $\mathcal{O}_k$ defined by

$$\overline{\mathcal{O}_k} = \cup_{j=1}^k \overline{\Omega_j}$$

are connected. Note that $\mathcal{O}_N = \mathbf{R}^2$.

Let u be an eigenfunction associated to some eigenvalue ω^2 of $A_\varepsilon(\beta)$,with $\omega^2 > c_\infty^2\beta^2$. We are going to show by induction that u vanishes in $\mathcal{O}_k$, $\forall k \in \{0, .., N\}$. We first prove that $u = O$ in $\mathcal{O}_0$. Since $\text{div}_\beta u = 0$ in $|\ x\ | \geq R$, it is clear that (use the identity $\text{rot}_\beta^*(\text{rot}_\beta) = \nabla_\beta(\text{div}_\beta) - \Delta + \beta^2$)

$$\Delta u + (\omega^2 - c_\infty^2\beta^2)u = 0 \quad \text{for } |\ x\ | \geq R\ .$$

As $u \in L^2(\mathbf{R}^2)$, we deduce from Rellich's theorem (see [28]) that $u = 0$ for $|\ x\ | \geq R$.

Now assume that u vanishes in $\mathcal{O}_{k-1}, k \geq 1$ then we claim that u is solution of

$$\text{rot}_\beta^*(\mu_k^{-1}\text{rot}_\beta u) = \varepsilon_k\omega^2 u \quad \text{in} \quad \mathcal{O}_k$$

Indeed, using (PR), for $v \in \mathcal{D}(\mathcal{O}_k)$, one has

$$\begin{cases} \displaystyle\int_{\mathcal{O}_k} \mu^{-1}\text{rot}_\beta u \cdot \text{rot}_\beta v\, dx = \int_{\Omega_k} \mu_k^{-1}\text{rot}_\beta u \cdot \text{rot}_\beta v\, dx \\ \displaystyle\int_{\mathcal{O}_k} \varepsilon u \cdot v\, dx = \int_{\Omega_k} \varepsilon_k u \cdot v\, dx\ . \end{cases}$$

To conclude, we shall assume here that $(\varepsilon_k, \mu_k) \in W^{2,\infty}(\mathbf{R}^2) \times W^{1,\infty}(\mathbf{R}^2)$. Otherwise it suffices to apply the forthcoming argument to $v = \mu^{-1}\text{rot}_\beta u$. Using the formulas ((e_1, e_2, e_3) denotes the canonical basis of $\mathbf{R}^3$)

$$\begin{cases} \text{rot}_\beta^*(\mu_k^{-1}\text{rot}_\beta u) = \mu_k^{-1}\text{rot}_\beta^*(\text{rot}_\beta u) - \mu_k^{-2}\vec{\text{rot}}\mu_k \text{rot}\boldsymbol{u} + \mu_k^{-2}\nabla\mu_k \cdot (\nabla u_3 - \beta\boldsymbol{u})e_3 \\ \text{rot}_\beta^*(\text{rot}_\beta u) = \nabla_\beta(\text{div}_\beta u) - \Delta u + \beta^2 u \\ \displaystyle\nabla_\beta(\text{div}_\beta u) = \nabla_\beta(\varepsilon_k^{-1}\text{div}_\beta(\varepsilon_k u)) - \sum_{j=1}^2 \left\{\varepsilon_k^{-1}\nabla\varepsilon_k \cdot \frac{\partial \boldsymbol{u}}{\partial x_j} + \frac{\partial}{\partial x_j}(\varepsilon_k^{-1}\nabla\varepsilon_k) \cdot \boldsymbol{u}\right\} e_j - \beta(\varepsilon_k^{-1}\nabla\varepsilon_k) \cdot \boldsymbol{u}\, e_3 \end{cases}$$

we see, as $\mathrm{div}_\beta(\varepsilon_k u) = 0$ in $\mathcal{O}_k$, that

$$\left\{\begin{array}{rcl} \Delta u &=& (\beta^2 - \omega^2\varepsilon_k\mu_k)u - \mu_k^{-1}\vec{\mathrm{rot}}\mu_k \mathrm{rot}\boldsymbol{u} + \mu_k^{-1}\nabla\mu_k.(\nabla u_3 - \beta\boldsymbol{u})e_3 \\ && \displaystyle -\sum_{j=1}^{2}\left\{\varepsilon_k^{-1}\nabla\varepsilon_k.\frac{\partial \boldsymbol{u}}{\partial x_j} + \frac{\partial}{\partial x_j}(\varepsilon_k^{-1}\nabla\varepsilon_k)\cdot\boldsymbol{u}\right\}e_j - \beta(\varepsilon_k^{-1}\nabla\varepsilon_k)\cdot\boldsymbol{u}\,e_3\,. \end{array}\right.$$

Then by elliptic regularity we deduce that $u \in H^2_{loc}(\mathcal{O}_k)^3$ and that

$$|\Delta u(x)| \leq C(|\nabla u(x)| + |u(x)|) \qquad \text{a.e. } x \text{ in } \mathcal{O}_k$$

where the constant C depends on $||\varepsilon_k||_{W^{2,\infty}}$ and $||\mu_k||_{W^{1,\infty}}$. We can then apply the unique continuation theorem of Section 1.3 to conclude that u vanishes identically in $\mathcal{O}_k$

<u>**Remark**</u> **3.1** *A priori $c_\infty^2\beta^2$ can be an eigenvalue. However one can prove that if u is an eigenfunction associated to the eigenvalue $c_\infty^2\beta^2$, then for $|x| \geq R$, it admits an expansion in the form*

(3.1)
$$u(x) = \sum_{n=2}^{+\infty}(u_n^0\cos n\theta + u_n^1\sin n\theta)r^{-n}$$

where $(x_1 = r\cos\theta,\ x_2 = r\sin\theta)$ ((r,θ) are the polar coordinates) and u_n^0 and u_n^1 are vectors in $\mathbf{R}^3$. We shall come back later on this point.

As a consequence of Lemma 3.1, if we except $c_\infty^2\beta^2$, eigenvalues of $A_\varepsilon(\beta)$ (or $A_\mu(\beta)$) belong to the discrete spectrum which moreover satisfies

(3.2)
$$\sigma_d(A_\varepsilon(\beta)) \subset (c_-^2\beta^2, c_\infty^2\beta^2)\,.$$

In particular any eigenvalue in the discrete spectrum of $A_\varepsilon(\beta)$ is necessarily strictly smaller than the lower bound $c_\infty^2\beta^2$ of its essential spectrum. This implies that all these eigenvalues can be characterized with the help of the Min-Max principle that we shall state below. For this we need to introduce the "Min-Max" associated to the operators $A_\varepsilon(\beta)$ and $A_\mu(\beta)$. We shall use the following notation:

- $\mathcal{V}_m^\varepsilon(\beta)$ is the set of m-dimensional subspaces of $V_\varepsilon(\beta)$
- $\mathcal{H}_m^\varepsilon(\beta)$ is the set of m-dimensional subspaces of $H_\varepsilon(\beta)$
- For any subset F of $H_\varepsilon(\beta)$, we set $F^\perp = \{u \in H_\varepsilon(\beta)\ /\ (u,v)_\varepsilon = 0, \quad \forall v \in F\}$.

Then for any $m \geq 1$, we introduce the real numbers given by the two equivalent formulas (cf. M. Reed and B. Simon [27], R. Courant and D. Hilbert [9]):

(3.3)
$$\begin{array}{rcl} s_m^\varepsilon(\beta) &=& \displaystyle\inf_{E\in\mathcal{V}_m^\varepsilon(\beta)}\ \sup_{u\in E\setminus\{0\}} \frac{a_\varepsilon(\beta\,;u,u)}{|u|_\varepsilon^2} \\ &=& \displaystyle\sup_{F\in\mathcal{H}_{m-1}^\varepsilon(\beta)}\ \inf_{u\in F^\perp\cap V_\varepsilon(\beta)} \frac{a_\varepsilon(\beta\,;u,u)}{|u|_\varepsilon^2} \end{array}$$

We also define, concerning the operator $A_\mu(\beta)$

$$
(3.4) \qquad \left| \begin{aligned} s^\mu_m(\beta) &= \inf_{E \in \mathcal{V}^\mu_m(\beta)} \sup_{u \in E \setminus \{0\}} \frac{a_\mu(\beta\,;u,u)}{|\,u\,|^2_\mu} \\ &= \sup_{F \in \mathcal{H}^\mu_{m-1}(\beta)} \inf_{u \in F^\perp \cap V_\mu(\beta)} \frac{a_\mu(\beta\,;u,u)}{|\,u\,|^2_\mu} \end{aligned} \right.
$$

where $\mathcal{V}^\mu_m(\beta)$, $\mathcal{H}^\mu_m(\beta)$ and $F^\perp$ are defined as above by simply replacing ε by μ. The Min-Max principle can be stated as follows for the operator $A_\varepsilon(\beta)$.

Theorem 3.1 *The sequence $s^\varepsilon_m(\beta)$ is nondecreasing and converges to $c^2_\infty\beta^2$. Moreover for each $m \geq 1$, one has the following alternative:*

(i) $s^\varepsilon_m(\beta) < c^2_\infty\beta^2$: in this case $A^\varepsilon(\beta)$ admits at least m eigenvalues strictly smaller than $c^2_\infty\beta^2$ and $\{s^\varepsilon_1(\beta), s^\varepsilon_2(\beta), \ldots, s^\varepsilon_m(\beta)\}$ are exactly the m first eigenvalues of $A_\varepsilon(\beta)$.

(ii) $s^\varepsilon_m = c^2_\infty\beta^2$: in this case $s^\varepsilon_j(\beta) = c^2_\infty\beta^2$ for any $j \geq m$ and $A_\varepsilon(\beta)$ has at most $(m-1)$ eigenvalues strictly smaller than $c^2_\infty\beta^2$.

Proof The theorem is nothing but the application of the general Min-Max principle to our particular case. For a proof of this principle, the reader is referred to [27] or [9].

As a consequence of Theorem 3.1 and Corollary 2.2, we have

Theorem 3.2

$$\forall m \geq 1, \quad s^\varepsilon_m(\beta) = s^\mu_m(\beta)\,(=_{def}\ s_m(\beta))$$

Proof This comes from the fact that $A_\varepsilon(\beta)$ and $A_\mu(\beta)$ have the same essential spectrum and the same eigenvalues. Indeed assume that $s^\varepsilon_m(\beta) < s^\mu_m(\beta)$. Necessarily, as $s^\mu_m(\beta) \leq c^2_\infty\beta^2$, $s^\varepsilon_m(\beta) < c^2_\infty\beta^2$ which means by Theorem 3.1 that $A_\varepsilon(\beta)$ admits at least m eigenvalues strictly smaller than $c^2_\infty\beta^2$. By Theorem 1.1, these eigenvalues are also eigenvalues of $A_\mu(\beta)$ which would mean that $A_\mu(\beta)$ has m eigenvalues strictly smaller than $s^\mu_m(\beta)$, which contradicts the Min-Max principle. Therefore $s^\varepsilon_m(\beta) \geq s^\mu_m(\beta)$. Inverting the rôles of ε and μ, $s^\mu_m(\beta) \geq s^\varepsilon_m(\beta)$, and thus $s^\varepsilon_m(\beta) = s^\mu_m(\beta)$.

Remark 3.2 *From Theorem 3.1, we deduce the rule for proving existence or nonexistence of eigenvalues.*

(i) If one can construct a subspace E of $V_\varepsilon(\beta)$ (or $V_\mu(\beta)$) such that $\dim E = m$ and

$$\forall u \in E \quad a_\varepsilon(\beta\,;u,u) - c^2_\infty\beta^2\,|\,u\,|^2_\varepsilon < 0$$

then $A_\varepsilon(\beta)$ admits at least m eigenvalues strictly smaller than $c^2_\infty\beta^2$.

(ii) If one can construct a subspace F of $H_\varepsilon(\beta)$ (or $H_\mu(\beta)$) such that $\dim F = m-1$ and

$$\forall u \in F^\perp \cap V_\varepsilon(\beta) \quad a_\varepsilon(\beta\,;u,u) - c^2_\infty\beta^2\,|\,u\,|^2_\varepsilon \geq 0$$

then $A_\varepsilon(\beta)$ admits at most $m-1$ eigenvalues strictly smaller than $c^2_\infty\beta^2$.

This rule is the rule we are going to apply for studying the point spectrum of $A_\varepsilon(\beta)$ (i.e. the guided waves). In the sequel we shall need a result concerning the regularity of the functions $\beta \to s_m^\varepsilon(\beta)$. The proof we give here is slightly more complicated than usual proofs for this type of result because of the fact that the space $V_\varepsilon(\beta)$ depends on β.

Theorem 3.3 *$\forall m \geq 1$, the functions $\beta \to s_m^\varepsilon(\beta)$, $\beta > 0$ are locally Lipschitz continuous.*

Proof To work in a space independent of β we first remark that $u = (\boldsymbol{u}, u_3) \in V_\varepsilon(\beta)$ if and only if $\tilde{u} = (\boldsymbol{u}, \tilde{u}_3) = (\boldsymbol{u}, \beta u_3) \in V_\varepsilon(1)$ and therefore that $s_m(\beta)$ is characterized by

$$s_m(\beta) = \inf_{E \in \mathcal{V}_m^\varepsilon(1)} \sup_{u \in E^*} \mathcal{R}_\varepsilon(\beta; u) \tag{3.5}$$

where we have set

$$\mathcal{R}_\varepsilon(\beta; u) = \frac{\tilde{a}_\varepsilon(\beta; u, u)}{|u|^2_{\varepsilon,\beta}} \tag{3.6}$$

with

$$\begin{cases} \tilde{a}_\varepsilon(\beta; u, u) = \displaystyle\int_{\mathbf{R}^2} \mu^{-1}(\ |\mathrm{rot}\boldsymbol{u}|^2 + |\frac{1}{\beta}\nabla u_3 - \beta \boldsymbol{u}|^2\ )\, dx \\ |u|^2_{\varepsilon,\beta} = |\boldsymbol{u}|^2_\varepsilon + \dfrac{1}{\beta^2}|u_3|^2_\varepsilon. \end{cases}$$

Now one computes that difference for any $(\beta, \beta') \in \mathbf{R}^{++} \times \mathbf{R}^{++}$:

$$\begin{aligned} \mathcal{R}_\varepsilon(\beta; u) - \mathcal{R}_\varepsilon(\beta'; u) &= (\beta^2 - \beta'^2)\frac{\|\boldsymbol{u}\|^2_\varepsilon \displaystyle\int_{\mathbf{R}^2} \mu^{-1}(\ |\boldsymbol{u}|^2 - \frac{|\nabla u_3|^2}{\beta^2\beta'^2}\ )\, dx}{|u|^2_{\varepsilon,\beta}|u|^2_{\varepsilon,\beta'}} \\ &+(\beta^2 - \beta'^2)\frac{|u_3|^2_\varepsilon \displaystyle\int_{\mathbf{R}^2} \mu^{-1}(\ |\mathrm{rot}\boldsymbol{u}|^2 + (\beta^2 + \beta'^2)|\boldsymbol{u}|^2 - 2\nabla u_3 \cdot \boldsymbol{u}\ )\, dx}{\beta^2\beta'^2|u|^2_{\varepsilon,\beta}|u|^2_{\varepsilon,\beta'}}. \end{aligned}$$

Remarking that we have the identity (deduced from a development of $\tilde{a}_\varepsilon(\beta)$)

$$\int_{\mathbf{R}^2} \mu^{-1}(\ |\mathrm{rot}\boldsymbol{u}|^2 + (\beta^2 + \beta'^2)|\boldsymbol{u}|^2 - 2\nabla u_3 \cdot \boldsymbol{u}\ )\, dx = \tilde{a}_\varepsilon(\beta'; u, u) + \int_{\mathbf{R}^2} \mu^{-1}\beta^2(|\boldsymbol{u}|^2 - \frac{|\nabla u_3|^2}{\beta^2\beta'^2})\, dx$$

we obtain

$$\frac{\mathcal{R}_\varepsilon(\beta; u) - \mathcal{R}_\varepsilon(\beta'; u)}{(\beta^2 - \beta'^2)} = \frac{\displaystyle\int_{\mathbf{R}^2} \varepsilon c^2(\ |\boldsymbol{u}|^2 - \frac{|\nabla u_3|^2}{\beta^2\beta'^2}\ )\, dx}{|u|^2_{\varepsilon,\beta}} + \mathcal{R}_\varepsilon(\beta'; u)\frac{|u_3|^2_\varepsilon}{\beta^2\beta'^2|u|^2_{\varepsilon,\beta}}.$$

Now we notice that $|u_3|^2_\varepsilon \leq \beta^2|u|^2_{\varepsilon,\beta}$, $\int_{\mathbf{R}^2} \varepsilon c^2|\boldsymbol{u}|^2 dx \leq c_+^2|u|^2_{\varepsilon,\beta}$ and that, using the coercivity of $a_\varepsilon(\beta; u, u)$ (cf. formula (2.1)),

$$\int_{\mathbf{R}^2} \varepsilon c^2|\nabla u_3|^2 dx \leq \frac{c_+^2}{c_-^2}\beta'^2\tilde{a}_\varepsilon(\beta;' u, u) = \frac{c_+^2}{c_-^2}\beta'^2\mathcal{R}_\varepsilon(\beta'; u)\,|u|^2_{\varepsilon,\beta'}.$$

So we obtain that

$$\left|\frac{\mathcal{R}_\varepsilon(\beta; u) - \mathcal{R}_\varepsilon(\beta'; u)}{(\beta^2 - \beta'^2)}\right| \leq c_+^2 + (\frac{1}{\beta'^2} + \frac{c_+^2}{c_-^2}\frac{|u|^2_{\varepsilon,\beta'}}{\beta^2|u|^2_{\varepsilon,\beta}})\mathcal{R}_\varepsilon(\beta'; u).$$

Finally as

$$\frac{|u|^2_{\varepsilon,\beta'}}{\beta^2|u|^2_{\varepsilon,\beta}} \leq Max(\frac{1}{\beta^2}, \frac{1}{\beta'^2}),$$

we end up with the inequality (since $\frac{1}{\beta'^2} \leq \frac{c_+^2}{c_-^2} Max(\frac{1}{\beta^2}, \frac{1}{\beta'^2})$)

$$\mathcal{R}_\varepsilon(\beta; u) \leq \mathcal{R}_\varepsilon(\beta'; u)\left\{1 + 2|\beta^2 - \beta'^2|\frac{c_+^2}{c_-^2} Max(\frac{1}{\beta^2}, \frac{1}{\beta'^2})\right\} + c_+^2|\beta^2 - \beta'^2|.$$

Taking the Min-Max over u of both members of this inequality we find that

$$s_m(\beta) \leq s_m(\beta')\left\{1 + 2|\beta^2 - \beta'^2|\frac{c_+^2}{c_-^2} Max(\frac{1}{\beta^2}, \frac{1}{\beta'^2})\right\} + |\beta^2 - \beta'^2|c_+^2.$$

As we can invert the rôles of β and β', we easily obtain

$$\frac{|s_m(\beta) - s_m(\beta')|}{|\beta - \beta'|} \leq (\beta + \beta')(c_+^2 + 2\frac{c_+^2}{c_-^2} Max(\frac{1}{\beta^2}, \frac{1}{\beta'^2}))$$

which completes the proof.

4 Study of the discrete spectrum

From now on, we shall denote by $N(\beta)$ the number of eigenvalues of $A_\varepsilon(\beta)$ in $(c_-^2\beta^2, c_\infty^2\beta^2)$. By the Min-Max principle, $N(\beta)$ is also characterized by ($N(\beta)$ can be a priori equal to $+\infty$)

$$\begin{cases} m \leq N(\beta) \implies s_m(\beta) < c_\infty^2\beta^2 \\ m > N(\beta) \implies s_m(\beta) = c_\infty^2\beta^2\ . \end{cases}$$

4.1 First existence result. Notion of the threshold

We immediately begin by a nonexistence result.

Lemma 4.1 *If $c(x)^2 \geq c_\infty^2$ almost everywhere, the discrete spectrum of $A_\varepsilon(\beta)$ (or $A_\mu(\beta)$) is empty.*

Proof In that case, the interval $(c_-^2\beta^2, c_\infty^2\beta^2)$ is empty.

A consequence of Lemma 4.1 is that in order to ensure the existence of isolated eigenvalues, it is necessary that the function $c(x)$ admits some where a strict minimum in the domain B_R. We shall see now that this condition is also (almost) sufficient to obtain an existence result at least for large values of β. The idea of the proof is that if $A_\varepsilon(\beta)$ admits an eigenvalue in $(c_-^2\beta^2, c_\infty^2\beta^2)$ then, because of (2.2), the quantity

$$c_\varepsilon(\beta\ ; u, u)\ =\ \beta^2\int_{\mathbb{R}^2}\varepsilon(c^2 - c_\infty^2)\mid \boldsymbol{u}\mid^2\ dx - 2\beta\int_{\mathbb{R}^2}\varepsilon(c^2 - c_\infty^2)\boldsymbol{u}.\nabla u_3\ dx$$

must be negative which will be true if u_3 can be chosen equal to 0 and $\boldsymbol{u}$ localized in some region where $c(x)^2 - c_\infty^2 < 0$. This is the idea of the proof which nevertheless must be modified because of the generalized divergence-free condition. This is the reason for which we shall be led to consider the following assumption:

$$(4.1)\quad \left|\begin{array}{l} \exists D_\ell \subset B_R,\ \text{where } D_\ell \text{ denotes a disk of radius } \ell \text{ such that} \\ \\ \text{(i)}\quad a.e.\ x \in D_\ell \quad c(x)^2 < c_\infty^2 - (\Delta c)^2, \quad \Delta c > 0 \\ \\ \text{(ii)}\quad \varepsilon(x) \in W^{2,\infty}(D_\ell) \text{ or } \mu(x) \in W^{2,\infty}(D_\ell)\,. \end{array}\right.$$

The reader will realize that in practice, assumption (4.1) differs very slightly from the more natural one:

$$(4.2)\qquad c_- < c_\infty\,.$$

For instance (4.1) and (4.2) are equivalent if ε and μ are piecewise $W^{2,\infty}$. We can also state our main existence result, which follows, using a weaker but more complicated assumption (see C. Poirier [26]).

Theorem 4.1 *Assume that (4.1) holds. Then for any $m \geq 1$, there exists $\beta_m \geq 0$ such that for any $\beta > \beta_m$, $A_\varepsilon(\beta)$ admits m eigenvalues strictly smaller than $c_\infty^2\beta^2$.*

Proof We shall assume that $\varepsilon(x) \in W^{2,\infty}(D_\ell)$. If not it suffices to reason with $a_\mu(\beta\ ;.,.)$ instead of $a_\varepsilon(\beta\ ;.,.)$. For $m \geq 1$, let us consider m 2D vector fields $\{\boldsymbol{u}_k^\ell, 1 \leq k \leq m\}$ satisfying

$$\left\{\ \boldsymbol{u}_k^\ell \in C_0^\infty(\mathbb{R}^2),\quad \text{supp } \boldsymbol{u}_k^\ell \subset D_\ell \quad \text{and} \quad \{\boldsymbol{u}_k^\ell, 1 \leq k \leq m\} \text{ are linearly independent.}\right.$$

Then define

$$\left|\ (u_3)_k^\ell \quad = \quad \frac{1}{\varepsilon\beta}\operatorname{div}(\varepsilon \boldsymbol{u}_k^\ell) \quad = \quad \frac{1}{\beta}\operatorname{div}(\boldsymbol{u}_k^\ell) + \frac{1}{\beta}\frac{\nabla\varepsilon}{\varepsilon}\cdot \boldsymbol{u}_k^\ell\,.\right.$$

As $\varepsilon \in W^{2,\infty}(D_\ell)$, it is clear that

$$\forall k \leq m \quad (u_3)_k^\ell \in H^1(\mathbb{R}^2), \quad \text{supp } (u_3)_k^\ell \subset D_\ell$$

and that

$$(4.3)\qquad \forall k \leq m \quad u_k^\ell = (\boldsymbol{u}_k^\ell, (u_3)_k^\ell) \ \in \ V_\varepsilon(\beta).$$

Now define the m dimensional subspace of $V_\varepsilon(\beta)$ generated by $\{u_1^\ell, \ldots, u_m^\ell\}$:

$$(4.4)\qquad E_m^\ell = [u_1^\ell, u_2^\ell, \ldots, u_m^\ell].$$

If $u = (\boldsymbol{u}, u_3) \in E_m^\ell$, we have, using (2.2)

$$\left|\begin{array}{rl} a_\varepsilon(\beta\ ;u,u) - c_\infty^2\beta^2 \mid u \mid_\varepsilon^2 = & \displaystyle\int_{\mathbb{R}^2} \varepsilon c^2(\mid \operatorname{rot}\boldsymbol{u} \mid^2 + \mid \nabla u_3 \mid^2)\,dx + c_\infty^2 \int_{\mathbb{R}^2} \varepsilon \mid \beta u_3 \mid^2\,dx \\ & \displaystyle -2\int_{\mathbb{R}^2} \varepsilon(c^2 - c_\infty^2)\nabla(\beta u_3)\cdot \boldsymbol{u}dx + \beta^2 \int_{\mathbb{R}^2} \varepsilon(c^2 - c_\infty^2) \mid \boldsymbol{u} \mid^2\,dx. \end{array}\right.$$

Using the fact that $\beta u_3 = \dfrac{1}{\varepsilon}\operatorname{div}(\varepsilon\boldsymbol{u})$ and part (i) of (4.1), we deduce that

$$(4.5)\quad \left|\begin{array}{l} \displaystyle a_\varepsilon(\beta\ ;u,u) - c_\infty^2\beta^2|u|_\varepsilon^2 \ \leq\ \frac{1}{\beta^2}\int_{\mathbb{R}^2} \varepsilon c^2 \mid \nabla(\frac{1}{\varepsilon}\operatorname{div}\varepsilon\boldsymbol{u})\mid^2\,dx + \int_{\mathbb{R}^2} \varepsilon c^2 \mid \operatorname{rot}\boldsymbol{u} \mid^2 dx \\ \displaystyle +\ \int_{\mathbb{R}^2} \varepsilon c_\infty^2 \mid \frac{1}{\varepsilon}\operatorname{div}(\varepsilon\boldsymbol{u}) \mid^2\,dx + 2\int_{\mathbb{R}^2} \varepsilon \mid c^2 - c_\infty^2 \mid\mid \boldsymbol{u}\cdot\nabla(\frac{1}{\varepsilon}\operatorname{div}(\varepsilon\boldsymbol{u}))\mid\,dx \\ \displaystyle -\ \beta^2(\Delta c)^2 \int_{\mathbb{R}^2} \varepsilon \mid \boldsymbol{u} \mid^2\,dx. \end{array}\right.$$

Denote by $M_1(m,\ell)$ and $M_2(m,\ell)$ the two positive constants defined by

$$
(4.6)\qquad \left|\begin{array}{rcl}
M_1(m,\ell) & = & \displaystyle\sup_{u\in E_m^\ell}\left(\frac{\displaystyle\int_{\mathbb{R}^2}\varepsilon c^2 \mid \nabla(\frac{1}{\varepsilon}\mathrm{div}(\varepsilon \boldsymbol{u}))\mid^2\, dx}{\displaystyle\int_{\mathbb{R}^2}\varepsilon\mid \boldsymbol{u}\mid^2\, dx}\right)\\
M_2(m,\ell) & = & \displaystyle\sup_{u\in E_m^\ell}\frac{Q_\varepsilon(u,u)}{\displaystyle\int_{\mathbb{R}^2}\varepsilon\mid \boldsymbol{u}\mid^2\, dx}
\end{array}\right.
$$

where we have defined

$$
Q_\varepsilon(u,u) = \int_{\mathbb{R}^2}\varepsilon c_\infty^2|\nabla(\frac{1}{\varepsilon}\mathrm{div}(\varepsilon\boldsymbol{u}))|^2 + 2\int_{\mathbb{R}^2}\varepsilon|c^2-c_\infty^2|\,|\boldsymbol{u}\cdot\nabla(\frac{1}{\varepsilon}\mathrm{div}(\varepsilon\boldsymbol{u}))|\,dx + \int_{\mathbb{R}^2}\varepsilon c^2|\mathrm{rot}\boldsymbol{u}|^2 dx\ .
$$

Note that $M_1(m,\ell)$ and $M_2(m,\ell)$ exist and are finite because E_m^ℓ has finite dimension. We deduce that

$$
a_\varepsilon(\beta\ ;u,u) - c_\infty^2\beta^2|u|_\varepsilon^2 \ \le\ \left(\frac{M_1(m,\ell)}{\beta^2} + M_2(m,\ell) - \beta^2\Delta c^2\right)\int_{\mathbb{R}^2}\varepsilon\mid \boldsymbol{u}\mid^2\, dx.
$$

If we choose

$$
(4.7)\qquad \beta\ >\ \frac{\sqrt{M_2+\sqrt{M_2^2+4M_1\Delta c^2}}}{\sqrt{2}\Delta c}
$$

we see that

$$
\forall u\in E_m^\ell,\quad u\neq 0\qquad a_\varepsilon(\beta\ ;u,u) - c_\infty^2\beta^2\mid u\mid_\varepsilon^2 <\ 0
$$

(note that because of the generalized divergence-free condition, $u \neq 0 \iff \boldsymbol{u}\neq 0$). This concludes the proof because of Theorem 3.1 (see section 3).

Theorem 4.1 points out a priori the existence of critical values of β which are thresholds for the apparition of eigenvalues in the discrete spectrum of $A_\varepsilon(\beta)$. This leads us to introduce the notion of upper thresholds

$$
(4.8)\qquad \begin{array}{rcl}\beta_m^* & = & \inf\{\ \beta_m>0\ \ /\ \ \forall\beta>\beta_m,\ A_\varepsilon(\beta)\ \text{admits at least}\ m\\ & & \qquad\text{eigenvalues strictly smaller than}\ c_\infty^2\beta^2\ \}.\end{array}
$$

By Min-Max principle, β_m^* is also characterized by

$$
(4.9)\qquad \beta_m^*\ =\ \inf\{\beta_m\ \ /\ \ s_m(\beta) < c_\infty^2\beta^2,\quad \forall\beta>\beta_m\}.
$$

A priori β_m^* can be equal to $+\infty$ (for instance if $c(x)$ is everywhere greater than c_∞) but Theorem 4.1 expresses that

$$
\text{assumption (4.1)} \implies \forall m\geq 1,\quad \beta_m^* < +\infty.
$$

In opposition to upper thresholds we also introduce the notion of lower threshold β_m^0 defined by

$$
(4.10)\qquad \begin{array}{rcl}\beta_m^0 & = & \sup\ \ \{\beta_m\ \ /\ \ \forall\beta<\beta_m,\ A_\varepsilon(\beta)\ \text{admits at most}\ m-1\\ & & \qquad\text{eigenvalues strictly smaller than}\ c_\infty^2\beta^2\ \}\end{array}
$$

or equivalently, using Min-Max principle,

$$\beta_m^0 = \sup \{\beta_m \quad / \quad \beta < \beta_m \implies s_m(\beta) = c_\infty^2 \beta^2\} \ . \tag{4.11}$$

It is immediate to verify that both sequences β_m^0 and β_m^* are nondecreasing and that, since the functions $s_m(\beta)$ are continuous (cf. Theorem 3.3)

$$\forall m \geq 1 \quad \beta_m^0 \ \leq \ \beta_m^* \ . \tag{4.12}$$

One can note that, from its definition, for $\beta < \beta_m^0$ the operator $A_\varepsilon(\beta)$ has at most $(m-1)$ eigenvalues and that the nonexistence result of Lemma 4.1 means that $\beta_1^0 = +\infty$ when $c(x) \geq c_\infty$ for a.e. $x \in \mathbb{R}^2$. It is thus clear that a lot of informations concerning guided waves can be expressed in terms of the properties of the thresholds β_m^0 and β_m^*. That is why our next section will be entirely devoted to the study of this two sequences.

5 Study of the thresholds

5.1 A case where $\beta_m^0 = \beta_m^*$

Here we consider a particular class of waveguides for which the propagation velocity is almost everywhere smaller than its value at infinity:

$$c(x) \leq c_\infty \quad a.e. \ x \in \mathbb{R}^2. \tag{5.1}$$

Under this assumption we have

Lemma 5.1 *If assumption (5.1) holds, for any $m \geq 1$, the function $\beta \to s_m(\beta) - c_\infty^2\beta^2$ is a decreasing function of β.*

Proof We adapt here the original idea of M. Reed and B. Simon [27], which has also been used in [7], for instance. First we remark that

$$\begin{aligned} s_m(\beta) - c_\infty^2\beta^2 &= \inf_{E \in V_m^\varepsilon(\beta)} \sup_{u \in E} \left(\frac{a_\varepsilon(\beta\ ; u, u) - c_\infty^2 \beta^2 \mid u \mid_\varepsilon^2}{\mid u \mid_\varepsilon^2} \right) \\ &= \inf_{E \in V_m^\varepsilon(\beta)} \sup_{u \in E} \left(\frac{p_\varepsilon(\beta\ ; u, u) + c_\varepsilon(\beta\ , u, u)}{\mid u \mid_\varepsilon^2} \right) . \end{aligned}$$

Now if $u = (\boldsymbol{u}, u_3)$ describes the space $V_\varepsilon(\beta)$, we note that $v = (\boldsymbol{u}, \beta u_3)$ describes the space $V_\varepsilon(1)$ and that

$$\left\{ \begin{aligned} c_\varepsilon(\beta\ ; u, u) + p_\varepsilon(\beta\ ; u, u) &= \frac{A_0(v)}{\beta^2} + A_1(v) + \beta^2 A_2(v) \\ \mid u \mid_\varepsilon^2 &= \frac{B_0(v)}{\beta^2} + B_1(v), \end{aligned} \right.$$

where we have set

$$\left\{ \begin{aligned} & A_0(v) = \int_{\mathbb{R}^2} \varepsilon c^2 \mid \nabla v_3 \mid^2 \ dx \\ & A_1(v) = \int_{\mathbb{R}^2} \varepsilon (c^2 \mid \operatorname{rot} \boldsymbol{v} \mid^2 + c_\infty^2 \mid v_3 \mid^2 - 2(c^2 - c_\infty^2) \nabla v_3 . \boldsymbol{v}) \ dx \\ & A_2(v) = \int_{\mathbb{R}^2} \varepsilon (c^2 - c_\infty^2) \mid \boldsymbol{v} \mid^2 \ dx \end{aligned} \right. \tag{5.2}$$

$$(5.3)\qquad \begin{cases} B_0(v) = \int_{\mathbf{R}^2} \varepsilon \mid v_3 \mid^2 \ dx \\ B_1(v) = \int_{\mathbf{R}^2} \varepsilon \mid \boldsymbol{v} \mid^2 \ dx. \end{cases}$$

Therefore we can write

$$(5.4)\qquad \begin{cases} s_m(\beta) - c_\infty^2\beta^2 &= \inf_{E\in\mathcal{V}_m^\varepsilon(1)} \sup_{v\in E} F(\beta, v) \\ F(\beta, v) &= \dfrac{A_0(v) + A_1(v)\beta^2 + A_2(v)\beta^4}{B_0(v) + B_1(v)\beta^2}. \end{cases}$$

We now use the fact that $s_m(\beta) - c_\infty^2\beta^2 \leq 0$ to remark that formula (5.4) does not change if we replace $F(\beta, v)$ by $G(\beta, v) = \inf(0, F(\beta, v))$. So we have

$$(5.5)\qquad s_m(\beta) - c_\infty^2\beta^2 = \inf_{E\in\mathcal{V}_m^\varepsilon(1)} \sup_{v\in E} G(\beta, v)\,.$$

As the set $\mathcal{V}_m^\varepsilon(1)$ does not depend on β, it suffices to prove that for any $v \in V_\varepsilon(1)$, $v \neq 0$, the function $\beta \to G(\beta, v)$ is nonincreasing. For this we need to study the variations of $\beta \to F(\beta, v)$. Now we note that only the function $A_1(v)$ has not a determined sign. Indeed we have by definition

$$(5.6)\qquad B_1(v) > 0, \quad B_0(v) \geq 0, \quad A_0(v) \geq 0 \quad (\forall v \neq 0)$$

while because of assumption (5.1), we also have

$$(5.7)\qquad A_2(v) \leq 0\,.$$

In order to be exhaustive, we have to distinguish 3 cases.

a) $v_3 = 0$

In such a case $A_0(v) = B_0(v) = 0$ and we can write

$$F(\beta, v) = \left(\frac{A_1(v)}{B_1(v)}\right) + \left(\frac{A_2(v)}{B_1(v)}\right)\beta^2\,.$$

As $\dfrac{A_2(v)}{B_1(v)} \leq 0$, $\beta \to F(\beta, v)$ is nonincreasing and so is $G(\beta, v)$.

b) $v_3 \neq 0$ and $A_2(v) = 0$

In such a case $A_0(v) > 0$ and $B_0(v) > 0$. Moreover

$$F(\beta, v) = \frac{A_0(v) + A_1(v)\beta^2}{B_0(v) + B_1(v)\beta^2}\,.$$

The function $\beta^2 \to F(\beta, v)$ is necessarily monotone. If $\dfrac{A_1(v)}{B_1(v)} \leq \dfrac{A_0(v)}{B_0(v)}$, then $\beta^2 \to F(\beta, v)$ is decreasing and therefore so is $\beta^2 \to G(\beta, v)$. And if $\dfrac{A_1(v)}{B_1(v)} > \dfrac{A_0(v)}{B_0(v)}$, then $\beta^2 \to F(\beta, v)$ is increasing and since $F(0, v) = \dfrac{A_0(v)}{B_0(v)} > 0$ the function $G(\beta, v)$ is equal to zero.

c) $v_3 \neq 0$ and $A_2(v) < 0$

In that case $F(\beta, v) \sim \dfrac{A_2(v)}{B_1(v)}\beta^2$ as $\beta \to +\infty$ with $\dfrac{A_2(v)}{B_1(v)} < 0$ and $F(0, v) = \dfrac{A_0(v)}{B_0(v)} > 0$.

Moreover one easily sees that the function $\dfrac{d}{d\beta^2}(F(\beta, v))$ has at most one zero. So that one necessarily meets one of the situations described by Figure 5.1, which proves once again that $G(\beta, v)$ is nonincreasing with respect to β and completes the proof of Lemma 5.1.

A direct consequence of Lemma 5.1 is the

Theorem 5.1 *If assumption (5.1) holds, then for any* $m \geq 1$, $\beta_m^0 = \beta_m^*$.

Remark 5.1 *We do not know any example for which* $\beta_m^0 < \beta_m^*$. *So we conjecture that the equality* $\beta_m^0 = \beta_m^*$ *is always true. However the proof of such a result remains an open question.*

5.2 Comparison results

In this section we consider two propagation media characterized respectively by $(\varepsilon_1(x), \mu_1(x))$ and $(\varepsilon_2(x), \mu_2(x))$. We suppose that these media are the same at infinity:

$$(5.8) \qquad \left\{ \varepsilon_1(x) = \varepsilon_2(x) = \varepsilon_\infty \quad \text{and} \quad \mu_1(x) = \mu_2(x) = \mu_\infty \quad \text{for } |x| \geq R \,. \right.$$

We denote by $s_m^1(\beta)$ and $s_m^2(\beta)$ the min-max respectively associated with these two media and by $N_1(\beta)$ and $N_2(\beta)$ the two corresponding numbers of eigenvalues strictly smaller than $c_\infty^2 \beta^2$. In the same way we shall use the notation $(\beta_m^{1,*}, \beta_m^{2,*})$ and $(\beta_m^{1,0}, \beta_m^{2,0})$ for the upper and lower thresholds. Our objective in this paragraph is to show that all these quantities can be compared provided that (ε_1, μ_1) and (ε_2, μ_2) can also be compared. Our precise result is the following

Theorem 5.2 *Assume that (5.8) holds and that*

$$(5.9) \qquad \left\{ \varepsilon_1(x) \leq \varepsilon_2(x) \quad \text{and} \quad \mu_1(x) \leq \mu_2(x) \quad \text{a.e. } x \in \mathbb{R}^2 \right.$$

then

$$(5.10) \qquad \forall m \geq 1, \quad \forall \beta > 0 \quad s_m^1(\beta) \geq s_m^2(\beta)$$

and consequently

$$(5.11) \qquad \left\{ \begin{array}{l} N_1(\beta) \leq N_2(\beta) \\[2mm] \beta_m^{1,*} \geq \beta_m^{2,*} \quad \text{and} \quad \beta_m^{1,0} \geq \beta_m^{2,0} \,. \end{array} \right.$$

Proof The proof is not direct and will use the two formulations in E and H. We will divide it into three parts.

(i) The result is true if $\varepsilon_1(x) = \varepsilon_2(x) = \varepsilon(x)$

Indeed if we set for $j = 1, 2$

$$a_\varepsilon^j(\beta\,; u, u) = \int_{\mathbb{R}^2} \mu_j^{-1} \mid \mathrm{rot}_\beta u \mid^2 \, dx \quad \forall u \in V_\varepsilon(\beta),$$

we clearly have

$$a_\varepsilon^1(\beta\ ; u,u) \ \geq\ a_\varepsilon^2(\beta\ ; u,u) \qquad \forall u \in V_\varepsilon(\beta).$$

Then as the space of test functions only depends on ε we can apply the Min-Max principle to show that

$$s_m^1(\beta) \ \geq\ s_m^2(\beta).$$

(ii) The result is true if $\mu_1(x) = \mu_2(x) = \mu(x)$

It suffices to invert the rôles of ε and μ and to use the H formulation.

(iii) Now assume that (5.9) holds and consider the intermediate medium defined by

$$\left\{ \ \tilde{\varepsilon}(x) = \varepsilon_1(x) \ \text{ and } \ \tilde{\mu}(x) = \mu_2(x)\ . \right.$$

Let us denote by $\tilde{s}_m(\beta)$ the corresponding min-max. Because of (i), we know that $s_m^1(\beta) \geq \tilde{s}_m(\beta)$ and because of (ii) we have $\tilde{s}_m(\beta) \geq s_m^2(\beta)$. Therefore $s_m^1(\beta) \geq s_m^2(\beta)$.

The other inequalities (5.11) derive directly from this one.

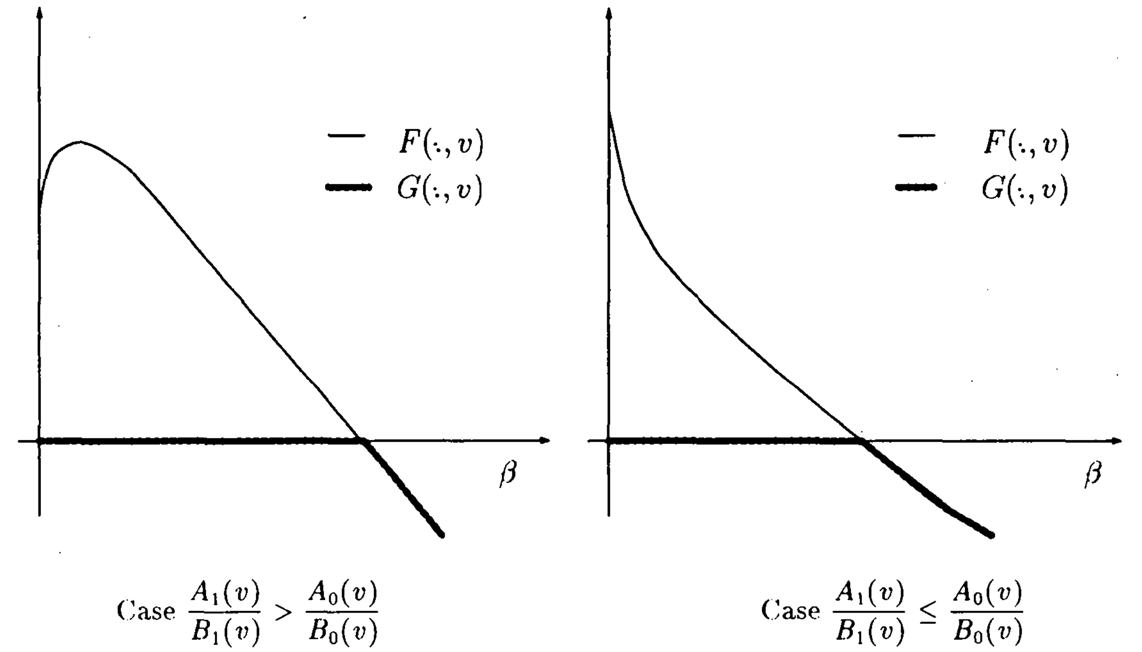

Figure 5.1: Curves of F and G for case c)

We will find this comparison result useful for extending some of our results to a more general class of media.

5.3 The threshold equation

Our objective in this section is to derive an equation satisfied by the thresholds. This equation will appear as a generalized eigenvalues equation. The idea is very simple, at least formally, and consists in passing to the limit in the eigenvalue problem for guided modes (under its variational form) when ω^2 tends to $c_\infty^2\beta^2$. The only, but essential, difficulty lies in the functional framework needed for the justification of the limit procedure. For this, we shall be led to introduce a weighted Sobolev type space of 2D vector fields. The same type of space occurs in the resolution of the Laplace equation in exterior domains (see A.S. Bonnet [7], J.C. Nedelec [23]). Let us introduce the weight function

$$\rho(x) = (1+ \mid x \mid^2)^{-1}(Log(2+ \mid x \mid^2))^{-2} > 0 \tag{5.12}$$

and define the space

$$H_\rho(\mathrm{rot}, \mathrm{div}_\varepsilon \,; \mathbb{R}^2) = \left\{ \boldsymbol{u} \in L^2_{loc}(\mathbb{R}^2)^2 \;/\; \rho^{\frac{1}{2}}\boldsymbol{u} \in L^2(\mathbb{R}^2)^2, (\mathrm{rot}\boldsymbol{u}, \mathrm{div}(\varepsilon\boldsymbol{u})) \in L^2(\mathbb{R}^2)^2 \right\}$$

which is an Hilbert space for the norm

$$|||u|||_\varepsilon^2 \;=\; \int_{\mathbb{R}^2} \mid \boldsymbol{u} \mid^2 \rho \; dx + \int_{\mathbb{R}^2} (\mid \mathrm{rot}\boldsymbol{u} \mid^2 + \mid \mathrm{div}(\varepsilon\boldsymbol{u}) \mid^2) \; dx \,. \tag{5.13}$$

The main property of this space lies in the following proposition, whose proof is given in Appendix A.

Proposition 5.1 *In the space $H_\rho(rot, div_\varepsilon; \mathbb{R}^2)$, the mapping*

$$\boldsymbol{u} \longrightarrow \left(\int_{B_R} \mid \boldsymbol{u} \mid^2 \; dx + \int_{\mathbb{R}^2} \mid (rot\boldsymbol{u} \mid^2 + \mid div(\varepsilon\boldsymbol{u}) \mid^2) \; dx \right)^{\frac{1}{2}}$$

is a norm equivalent to the norm $|||.|||_\varepsilon$. Moreover the mapping $\boldsymbol{u} \to \boldsymbol{u} \mid_{B_R}$ is compact from $H_\rho(rot, div_\varepsilon \,; \mathbf{R}^2)$ into $L^2(B_R)^2$.

Remark 5.2
- *Note that $H_\rho(rot, div_\varepsilon \,; \mathbb{R}^2)$ is a greater space than $H(rot, div_\varepsilon \,; \mathbf{R}^2)$ (see section 2, proposition 2.2). The functions of these two spaces only differ by their behaviour at infinity. However they have the same local regularity and that is why local compactness properties are conserved (compare proposition 2.2 and proposition 5.1).*
- *It can be shown, using an adaptation of the truncation procedure used by J. Giroire in [16], that the embedding $H(rot, div_\varepsilon \,; \mathbf{R}^2) \hookrightarrow H_\rho(rot, div_\varepsilon \,; \mathbb{R}^2)$ is continuous and dense. In fact one can prove that functions of $H_\rho(rot, div_\varepsilon \,; \mathbb{R}^2)$ which have compact support (and thus belong to $H(rot, div_\varepsilon \,; \mathbb{R}^2)$) are dense in $H_\rho(rot, div_\varepsilon \,; \mathbb{R}^2)$ (see also Lemma 5.2 below).*

We shall also need to introduce the space of 3D vector fields

$$\left\{ \begin{array}{l} H_\rho(\mathrm{rot}, \mathbb{R}^2) \;\times\; H^1(\mathbb{R}^2) \\ \\ \text{where} \quad H_\rho(\mathrm{rot}, \mathbb{R}^2) \;=\; \{\, \boldsymbol{u} \in L^2_{loc}(\mathbb{R}^2)^2 \,,\; \rho^{\frac{1}{2}}\boldsymbol{u} \in L^2(\mathbb{R}^2)^2 \,,\; \mathrm{rot}\boldsymbol{u} \in L^2(\mathbb{R}^2) \,\}. \end{array} \right. \tag{5.14}$$

which is an Hilbert space for the norm

$$\| \; u \; \|^2_{H_\rho(\mathrm{rot},\mathbb{R}^2)\times H^1(\mathbb{R}^2)} \;=\; \int_{\mathbb{R}^2} (\rho \mid \boldsymbol{u} \mid^2 + \mid \mathrm{rot}\boldsymbol{u} \mid^2) \, dx + \int_{\mathbb{R}^2} (\mid u_3 \mid^2 + \mid \nabla u_3 \mid^2) \, dx. \tag{5.15}$$

Then the space (defined for $\beta > 0$)

$$V_{\rho,\varepsilon}(\beta) \;=\; \{u \in H_\rho(\mathrm{rot}, \mathbb{R}^2) \times H^1(\mathbb{R}^2) \;/\; \mathrm{div}_\beta(\varepsilon u) \;=\; 0\} \tag{5.16}$$

is a closed subspace of $H_\rho(\mathrm{rot}, \mathbb{R}^2) \times H^1(\mathbb{R}^2)$ and thus an Hilbert space if we equip it with the norm $\| \cdot \|_{H_\rho(\mathrm{rot},\mathbb{R}^2)\times H^1(\mathbb{R}^2)}$. A direct consequence of Proposition 5.1 is

Corollary 5.1 *The mapping*

$$u \to \left(\int_{B_R} |\, \boldsymbol{u} \,|^2 \; dx + \int_{\mathbb{R}^2} (|\, rot\boldsymbol{u} \,|^2 + |\, u_3 \,|^2 + |\, \nabla u_3 \,|^2) \; dx \right)^{1/2}$$

is a norm on the space $V_{\rho,\varepsilon}(\beta)$ equivalent to the norm $\| \cdot \|_{H_\rho(rot,\mathbb{R}^2)\times H^1(\mathbb{R}^2)}$ and the mapping $u \to u\,|_{B_R}$ is compact from $V_{\rho,\varepsilon}(\beta)$ into $L^2(B_R)^3$.

We can now define our threshold equation. First we remark that the two bilinear forms, previously introduced in Proposition 2.1

$$\left| \begin{array}{lcl} p_\varepsilon(\beta\,; u, v) & = & \displaystyle\int_{\mathbb{R}^2} \varepsilon c^2(\text{rot}\boldsymbol{u}\cdot\text{rot}\boldsymbol{v} + \nabla u_3\cdot\nabla v_3)\; dx + c_\infty^2\beta^2 \int_{\mathbb{R}^2} \varepsilon u_3 v_3 \; dx \\ c_\varepsilon(\beta\,; u, v) & = & \displaystyle\int_{\mathbb{R}^2} \varepsilon(c^2 - c_\infty^2)\,(\beta^2 \boldsymbol{u}\cdot\boldsymbol{v} - \beta(\nabla u_3\cdot\boldsymbol{v} + \nabla v_3\cdot\boldsymbol{u}))\; dx \end{array} \right.$$

are defined and continuous in the space $V_{\rho,\varepsilon}(\beta)$.

Then we shall say that $\beta > 0$ satisfies **the threshold equation** if it is solution of the following problem:

$$(5.17) \qquad \left\{ \begin{array}{l} \text{Find } \beta > 0 \;/\; \exists u \neq 0 \in V_{\rho,\varepsilon}(\beta) \\ \\ \forall v \in V_{\rho,\varepsilon}(\beta) \quad p_\varepsilon(\beta\,; u, v) + c_\varepsilon(\beta\,; u, v) \;=\; 0. \end{array} \right.$$

Before stating the main result of this section, we need a technical theorem concerning the functional space $V_{\rho,\varepsilon}(\beta)$. The proof of this result is given in Appendix B.

Lemma 5.2 *Let $V_{\varepsilon,c}(\beta)$ be the subspace of $V_\varepsilon(\beta)$ (and thus of $V_{\rho,\varepsilon}(\beta)$) made of compactly supported functions, then $V_{\varepsilon,c}(\beta)$ is dense in $V_{\rho,\varepsilon}(\beta)$*

We can now state the main result of this section.

Theorem 5.3 *If β is a threshold such that $0 < \beta < +\infty$, then it is solution of the threshold equation (5.17).*

Proof 1) By definition of the upper and lower thresholds, any of them can be characterized by

$$(5.18) \qquad \beta = \lim_{p\to+\infty} \beta_p$$

where the decreasing sequence $(\beta_p)_{p\geq 1}$ is such that there exists for each $p \geq 1$ a function u_p in $D(A_\varepsilon(\beta_p))$, $u_p \neq 0$, and a real $\omega_p^2 \in (c_-^2\beta_p^2, c_\infty^2\beta_p^2)$ which satisfy

$$\left\{ \begin{array}{lcl} A_\varepsilon(\beta_p)\, u_p & = & \omega_p^2\, u_p \\ \\ \displaystyle\lim_{p\to+\infty} \omega_p^2 & = & c_\infty^2\beta^2. \end{array} \right.$$

We thus have $u_p \in V_\varepsilon(\beta_p)$ and

$$(5.19) \qquad \forall v \in V_\varepsilon(\beta_p) \quad a_\varepsilon(\beta_p\,; u_p, v) \;=\; \omega_p^2\,(u_p, v)_\varepsilon.$$

In particular as $\omega_p^2 < c_\infty^2\beta_p^2$, we have

$$a_\varepsilon(\beta_p\ ; u_p, u_p) - c_\infty^2\beta_p^2 \mid u_p \mid_\varepsilon^2 < 0,$$

that is to say

$$p_\varepsilon(\beta_p\ ; u_p, u_p) + c_\varepsilon(\beta_p\ ; u_p, u_p) < 0\ . \tag{5.20}$$

If $u_p = (\boldsymbol{u}_p, u_{3,p})$, the vector field $\tilde{u}_p = (\boldsymbol{u}_p, \tilde{u}_{3,p})$ with $\tilde{u}_{3,p} = \beta_p u_{3,p}$ belongs to the fixed space (with respect to p) $V_\varepsilon(1)$. Moreover

$$(5.21)\quad \left|\begin{array}{l} p_\varepsilon(\beta_p\ ; u_p, u_p) = \displaystyle\int_{\mathbf{R}^2} \varepsilon c^2\ (\mid \mathrm{rot}\boldsymbol{u}_p \mid^2 + \frac{1}{\beta_p^2} \mid \nabla\tilde{u}_{3,p} \mid^2)\ dx + c_\infty^2 \int_{\mathbf{R}^2} \varepsilon \mid \tilde{u}_{3,p} \mid^2\ dx \\ c_\varepsilon(\beta_p\ ; u_p, u_p) = \beta_p^2 \displaystyle\int_{B_R} \varepsilon(c^2 - c_\infty^2) \mid \boldsymbol{u}_p \mid^2\ dx - 2\int_{B_R} \varepsilon(c^2 - c_\infty^2)\nabla\tilde{u}_{3,p} \cdot \boldsymbol{u}_p\ dx. \end{array}\right.$$

Note that the strict inequality (5.20) implies that the restriction of $\boldsymbol{u}_p$ to B_R is not identically zero. Therefore, as u_p is defined up to a multiplicative constant (it is an eigenvector), it can be normalized such as

$$\int_{B_R} \varepsilon \mid \boldsymbol{u}_p \mid^2\ dx\ =\ 1. \tag{5.22}$$

Let $\eta(x) > 0$, using Young's inequality we get

$$\left|\begin{array}{rcl} c_\varepsilon(\beta_p\ ; u_p, u_p) & \geq & \beta_p^2 \displaystyle\int_{\mathbf{R}^2} \varepsilon(c^2 - c_\infty^2) \mid \boldsymbol{u}_p \mid^2\ dx - \int_{\mathbf{R}^2} \frac{\eta}{\beta_p^2} \mid \nabla\tilde{u}_{3,p} \mid^2\ dx \\ & & - \displaystyle\int_{\mathbf{R}^2} \frac{\varepsilon^2(c^2 - c_\infty^2)^2}{\eta} \beta_p^2 \mid \boldsymbol{u}_p \mid^2\ dx\ . \end{array}\right.$$

Let us choose $\eta = \dfrac{1}{2}\varepsilon c^2$, we get

$$(5.23)\quad \left|\begin{array}{rcl} c_\varepsilon(\beta_p\ ; u_p, u_p) & \geq & \beta_p^2 \displaystyle\int_{\mathbf{R}^2} \varepsilon(c^2 - c_\infty^2)[1 - 2(c^2 - c_\infty^2)/c^2] \mid \boldsymbol{u}_p \mid^2\ dx \\ & & - \dfrac{1}{2\beta_p^2} \displaystyle\int_{\mathbf{R}^2} \varepsilon c^2 \mid \nabla\tilde{u}_{3,p} \mid^2\ dx\ . \end{array}\right.$$

Plugging (5.23) into (5.20) we obtain that

$$(5.24)\quad \left|\begin{array}{l} \displaystyle\int_{\mathbf{R}^2} \mid \mathrm{rot}\boldsymbol{u}_p \mid^2 \varepsilon c^2\ dx + \frac{1}{2\beta_p^2} \int_{\mathbf{R}^2} \mid \nabla\tilde{u}_{3,p} \mid^2 \varepsilon c^2\ dx \\ \qquad\qquad + c_\infty^2 \displaystyle\int_{\mathbf{R}^2} \varepsilon \mid \tilde{u}_{3,p} \mid^2\ dx\ \leq\ C\ \beta_p^2 \end{array}\right.$$

(the constant C is $\| (c^2 - c_\infty^2)(2\dfrac{c_\infty^2}{c^2} - 1) \|_{L^\infty}$). Consequently, as β_p and $\dfrac{1}{\beta_p}$ are bounded and as $\mathrm{div}(\varepsilon\boldsymbol{u}_p) = \varepsilon\tilde{u}_{3,p}$ (since $\tilde{u}_p$ belongs to $V_{p,\varepsilon}(1)$), we deduce that

$$\tilde{u}_p \text{ is bounded in } V_{p,\varepsilon}(1). \tag{5.25}$$

2) By compactness one can extract a subsequence, still denoted by $\tilde{u}_p$ such that

$$(5.26)\quad \left\{\begin{array}{lll} \tilde{u}_p \rightharpoonup \tilde{u} & \text{weakly in} & V_{p,\varepsilon}(1) \\ \tilde{u}_p \to \tilde{u} & \text{strongly in} & L^2(B_R)^3\ . \end{array}\right.$$

The variational eigenvalue problem (5.19) can be rewritten, using $\tilde{u}_p$ instead of u_p (and $\tilde{v} = (\boldsymbol{v}, \tilde{v}_3) \in V_\varepsilon(1)$ instead of $v \in V_\varepsilon(\beta_p)$) as follows:

$$\int_{\mathbf{R}^2} \varepsilon c^2 \mathrm{rot}\boldsymbol{u}_p \cdot \mathrm{rot}\boldsymbol{v}\, dx + \frac{1}{\beta_p^2}\int_{\mathbf{R}^2} \varepsilon c^2 \nabla\tilde{u}_{3p}\cdot\nabla\tilde{v}_3\, dx + c_\infty^2 \int_{\mathbf{R}^2}\varepsilon\tilde{u}_{3p}\tilde{v}_3\, dx$$
$$+\int_{B_R}\varepsilon(c^2-c_\infty^2)\left(\beta_p^2\boldsymbol{u}_p\cdot\boldsymbol{v} - \nabla\tilde{u}_{3p}\cdot\boldsymbol{v} - \nabla\tilde{v}_3\cdot\boldsymbol{u}_p\right)\, dx$$
$$= (\omega_p^2 - \beta_p^2 c_\infty^2)\int_{\mathbf{R}^2}(\boldsymbol{u}_p\cdot\boldsymbol{v} + \frac{1}{\beta_p^2}\tilde{u}_{3p}\tilde{v}_3)\, dx.$$

This holds in particular if $\tilde{v} \in V_{\varepsilon,c}(1)$ ($\tilde{v}$ has a compact support). In such a case, we can pass to the limit when p goes to infinity since in such a case the weight function ρ does not play any rôle. So we deduce that for any $v \in V_{\varepsilon,c}(1)$

$$\int_{\mathbf{R}^2}\varepsilon c^2 \mathrm{rot}\boldsymbol{u}\mathrm{rot}\boldsymbol{v}\, dx + \frac{1}{\beta^2}\int_{\mathbf{R}^2}\varepsilon c^2\nabla\tilde{u}_3\cdot\nabla\tilde{v}_3\, dx + c_\infty^2\int_{\mathbf{R}^2}\varepsilon\tilde{u}_3\tilde{v}_3\, dx$$
$$+\int_{B_R}\varepsilon(c^2-c_\infty^2)\left(\beta^2\boldsymbol{u}\cdot\boldsymbol{v} - \nabla\tilde{u}_3\cdot\boldsymbol{v} - \nabla\tilde{v}_3\cdot\boldsymbol{u}\right)\, dx = 0\,. \tag{5.27}$$

Using again the change of unknown function $\tilde{u} = (\boldsymbol{u}, \tilde{u}_3) \to u = (\boldsymbol{u}, \frac{1}{\beta}\tilde{u}_3)$ (and the same for the test function $\tilde{v} \to v$), we remark that (5.27) is nothing but

$$p_\varepsilon(\beta\,; u, v) + c_\varepsilon(\beta\,; u, v) = 0 \quad \forall v \in V_{\varepsilon,c}(\beta) \tag{5.28}$$

As $V_{\varepsilon,c}(\beta)$ is dense in $V_{\rho,\varepsilon}(\beta)$ (cf. lemma 5.2) and as $p_\varepsilon(\beta\,;.,.)$ and $c_\varepsilon(\beta\,;.,.)$ are continuous in $V_{\rho,\varepsilon}(\beta)$, (5.28) also holds for any v in $V_{\rho,\varepsilon}(\beta)$. To conclude it remains to prove that u is nonzero. This comes immediately by passing to the limit in (5.22), using the strong convergence (5.26).

Remark 5.3 • *Note that we have proven that the thresholds are solutions of the threshold equation but we don't know if the converse is true. However such a type of result has been proven for scalar waveguides in [7].*

- *In fact we proved in Theorem 5.3 that any number β which is the limit of a sequence β_p for which the operator $A_\varepsilon(\beta_p)$ admits an eigenvalue ω_p^2 tending to $\beta^2 c_\infty^2$ when $p \to +\infty$ is solution of the threshold equation. Both upper and lower thresholds β_m^* and β_m^0 have this property but they could exist other β's, in particular if there exists $m \geq 1$ for which $\beta_m^0 < \beta_m^*$.*

- *In section 3, we evoked the possibility for $c_\infty^2\beta^2$ to be an eigenvalue. We can now state a necessary and sufficient condition for this which is*

$$\begin{cases} (i) & \beta \text{ is a solution of the threshold equation} \\ (ii) & \text{the corresponding generalized eigenvalue } u \text{ belongs to } L^2(\mathbf{R}^2)^3 \end{cases}$$

- *Note that the threshold equation (5.17) only concerns the strictly positive threshold. The question of the threshold* 0 *will be treated in section 5.6.*

5.4 The third lower threshold β_3^0 is strictly positive

We are going to prove in this section that for low values of β, at most two modes can propagate. This can be stated as follows.

Theorem 5.4 *The third lower threshold β_3^0 is strictly positive.*

Before giving the proof of Theorem 5.4, we need an intermediate result which we shall also use in section 5.6. This result concerns a particular subspace of $H_\rho(\text{rot}\,;\text{div}_\varepsilon\,;\mathbb{R}^2)$, which is

$$\mathbf{P}_\varepsilon \;=\; \{u \in H_\rho(\text{rot}\,;\text{div}_\varepsilon;\mathbb{R}^2)\;/\;\text{rot}\boldsymbol{u} \;=\; \text{div}(\varepsilon\boldsymbol{u}) \;=\; 0\}\,. \tag{5.29}$$

In order to characterize the space $\mathbf{P}_\varepsilon$ we shall use a nonlocal operator T acting on functions defined on $\Gamma_R = \partial B_R$. We shall use polar coordinates and the polar angle θ for the parameterization of Γ_R. We can then characterize the Sobolev spaces $H^s(\Gamma_R)$ by Fourier series as follows:

$$\left\{\begin{array}{l} g(\theta) = \sum\limits_{n=0}^{+\infty} g_n \cos n\theta + \sum\limits_{n=1}^{+\infty} \tilde{g}_n \sin n\theta \quad \in \quad H^s(\Gamma_R) \\ \Longleftrightarrow \quad \sum\limits_{n=1}^{+\infty} (1+n^2)^s(\mid g_n\mid^2 + \mid \tilde{g}_n \mid^2) < +\infty. \end{array}\right.$$

Our operator T is defined and continuous from $H^{\frac{1}{2}}(\Gamma_R)$ into $H^{-\frac{1}{2}}(\Gamma_R)$ as follows:

$$\left| \begin{array}{lcl} g(\theta) &=& \sum_{n=0}^{+\infty} g_n \cos n\theta + \sum_{n=1}^{+\infty} \tilde{g}_n \sin n\theta \quad \Longrightarrow \\ Tg &=& \dfrac{1}{R}\sum\limits_{n=1}^{+\infty} n\,(g_n \cos n\theta + \tilde{g}_n \sin n\theta). \end{array}\right. \tag{5.30}$$

We have the

Lemma 5.3 *The space $\mathbf{P}_\varepsilon$ is of dimension 2. It is generated by the functions $\boldsymbol{u}^1 = \nabla\varphi^1$ and $\boldsymbol{u}^2 = \nabla\varphi^2$ where the functions φ^1 and φ^2 are defined as follows:*

(i) in B_R, φ_1 and φ_2 are the unique solutions of the problems

$$\begin{array}{l} \left\{\begin{array}{ll} \varphi^1 \in H^1(B_R), \quad \int_{\Gamma_R} \varphi_1 \, d\sigma = 0 & \\ div(\varepsilon\nabla\varphi^1) = 0 & in\ B_R \\ \dfrac{\partial\varphi^1}{\partial n} + T\varphi^1 = 2\cos\theta & on\ \Gamma_R \end{array}\right. \\ \left\{\begin{array}{ll} \varphi^2 \in H^1(B_R), \quad \int_{\Gamma_R} \varphi_2 \, d\sigma = 0 & \\ div(\varepsilon\nabla\varphi^2) = 0 & in\ B_R \\ \dfrac{\partial\varphi^2}{\partial n} + T\varphi^2 = 2\sin\theta & on\ \Gamma_R \end{array}\right. \end{array} \tag{5.31}$$

(ii) in $\mathbb{R}^2 \backslash B_R$, φ^1 and φ^2 are given in polar coordinates (r, θ) by

$$
(5.32) \quad \left\{ \begin{array}{rcl}
\varphi^1(r) & = & \left\{ \varphi_1^1(R)\frac{R}{r} + r(1 - \frac{R^2}{r^2}) \right\} \cos\theta + \tilde{\varphi}_1^1(R)\frac{R}{r}\sin\theta \\
& & + \sum_{n=2}^{+\infty} \{\varphi_n^1(R)\cos n\theta + \tilde{\varphi}_n^1(R)\sin n\theta\} (\frac{r}{R})^{-n} \\
\varphi^2(r) & = & \left\{ \tilde{\varphi}_1^2(R)\frac{R}{r} + r(1 - \frac{R^2}{r^2}) \right\} \sin\theta + \varphi_1^2(R)\frac{R}{r}\cos\theta \\
& & + \sum_{n=2}^{+\infty} \{\varphi_n^2(R)\cos n\theta + \tilde{\varphi}_n^2(R)\sin n\theta\} (\frac{r}{R})^{-n}
\end{array} \right.
$$

where $\{\varphi_n^j(R)\}$ and $\{\tilde{\varphi}_n^j(R)\}$are the Fourier coefficients of $\varphi^j \mid_{\Gamma_R}, j = 1, 2$ (which are known because of (i)).

Proof Let u be an element of $\mathbf{P}_\varepsilon$. As $u \in L^2_{loc}(\mathbb{R}^2)$ and rot$u = 0$, we know from Poincaré 's lemma (see [10],[15]) that there exists a function φ in $H^1_{loc}(\mathbb{R}^2)$ such that $u = \nabla\varphi$. Moreover as this function is defined up to an additive constant we can impose the condition

$$
(5.33) \qquad \int_{\Gamma_R} \varphi \, d\sigma = 0 .
$$

From the condition div$(\varepsilon u) = 0$ we deduce that div$(\varepsilon\nabla\varphi) = 0$. From now on we shall use polar coordinates:

$$
\left(x_1 = r\cos\theta \quad , \quad x_2 = r\sin\theta \right)
$$

and a decomposition of φ in Fourier series with respect to θ:

$$
(5.34) \qquad \varphi(r, \theta) = \sum_{n=0}^{+\infty} \varphi_n(r)\cos n\theta + \sum_{n=1}^{+\infty} \tilde{\varphi}_n(r)\sin n\theta .
$$

By Plancherel's theorem we have

$$
\left| \begin{array}{l}
\int_0^{2\pi} \mid \varphi(r,\theta) \mid^2 \, d\theta = \pi \sum_{n=1}^{+\infty} (\mid \varphi_n(r) \mid^2 + \mid \tilde{\varphi}_n(r) \mid^2) + 2\pi \mid \varphi_0(r) \mid^2 \\
\int_0^{2\pi} \mid \frac{\partial\varphi}{\partial r}(r,\theta) \mid^2 \, d\theta = \pi \sum_{n=1}^{+\infty} (\mid \varphi_n'(r) \mid^2 + \mid \tilde{\varphi}'_n(r) \mid^2) + 2\pi \mid \varphi_0'(r) \mid^2 .
\end{array} \right.
$$

The condition $\int_{\mathbb{R}^2} \rho \mid u \mid^2 \, dx = \int_{\mathbb{R}^2} \rho \mid \nabla\varphi \mid^2 \, dx < +\infty$ implies that

$$
(5.35) \qquad \forall n \geq 1 \quad \int_2^{+\infty} \frac{\mid \varphi_n'(r) \mid^2 + \mid \tilde{\varphi}'_n(r) \mid^2}{r(Log\ r)^2} \, dr < +\infty
$$

$$
(5.36) \qquad \int_2^{+\infty} \frac{\mid \varphi_0'(r) \mid^2}{r(Log\ r)^2} \, dr < +\infty.
$$

For $r \geq R$, we have $\Delta\varphi = 0$ which gives the ordinary differential equation

$$
(5.37) \quad \left\{ \begin{array}{ll}
\varphi_n''(r) + \frac{1}{r}\varphi_n'(r) - \frac{n^2}{r^2}\varphi_n(r) = 0 & n = 0, 1, 2, \ldots \\
\tilde{\varphi}''_n(r) + \frac{1}{r}\tilde{\varphi}'_n(r) - \frac{n^2}{r^2}\tilde{\varphi}_n(r) = 0 & n = 0, 1, 2, \ldots
\end{array} \right.
$$

For $n \geq 1$, the general solution of these linear differential equations is a linear combination of r^n and r^{-n}. The condition (5.35) leads us to eliminate the function r^n for $n \geq 2$. Therefore we can write

$$
\left\{
\begin{array}{ll}
\varphi_n(r) = \varphi_n(R)(\frac{r}{R})^{-n} & \text{for } n \geq 2, r \geq R \\
\tilde{\varphi}_n(r) = \tilde{\varphi}_n(R)(\frac{r}{R})^{-n} & \text{for } n \geq 2, r \geq R
\end{array}
\right.
\tag{5.38}
$$

and

$$
\left\{
\begin{array}{l}
\varphi_1(r) = a_1 r + b_1 r^{-1} \\
\tilde{\varphi}_1(r) = \tilde{a}_1 r + \tilde{b}_1 r^{-1} .
\end{array}
\right.
\tag{5.39}
$$

For $n = 0$, we have

$$
\varphi_0(r) = a_0 Log\ r + b_0 .
\tag{5.40}
$$

Note that the fact that $\mathrm{div}(\varepsilon\nabla\varphi) = 0$ in B_R implies that (just integrate this equation over B_R)

$$
\int_{\Gamma_R} (\frac{\partial\varphi}{\partial r})\, d\sigma = 0.
\tag{5.41}
$$

Both equations (5.33) and (5.41) imply that

$$
\varphi_0(R) = \varphi_0'(R) = 0.
\tag{5.42}
$$

This yields $a_0 = b_0 = 0$ and thus

$$
\varphi_0(r) = 0 \quad \text{for } r \geq R.
\tag{5.43}
$$

Formulas (5.38) lead to the natural relation between $(\varphi_n'(R), \tilde{\varphi}_n'(R))$ and $(\varphi_n(R), \tilde{\varphi}_n(R))$ when $n \geq 2$

$$
\left\{
\begin{array}{ll}
\varphi_n'(R) + \frac{n}{R}\varphi_n(R) = 0 & n \geq 2 \\
\tilde{\varphi}_n'(R) + \frac{n}{R}\tilde{\varphi}_n(R) = 0 & n \geq 2.
\end{array}
\right.
\tag{5.44}
$$

In what concerns φ_1 and $\tilde{\varphi}_1$, one has

$$
\left\{
\begin{array}{l}
\varphi_1'(R) = a_1 - \frac{b_1}{R^2} \\
\varphi_1(R) = a_1 R + \frac{b_1}{R}
\end{array}
\right.
\quad \text{and} \quad
\begin{array}{l}
\tilde{\varphi}_1'(R) = \tilde{a}_1 - \frac{\tilde{b}_1}{R^2} \\
\tilde{\varphi}_1(R) = \tilde{a}_1 R + \frac{\tilde{b}_1}{R}.
\end{array}
$$

Eliminating b_1 and $\tilde{b}_1$, we get

$$
\left\{
\begin{array}{lcl}
\varphi_1'(R) + \frac{1}{R}\varphi_1(R) & = & 2a_1 \\
\tilde{\varphi}_1'(R) + \frac{1}{R}\varphi_1(R) & = & 2\tilde{a}_1.
\end{array}
\right.
\tag{5.45}
$$

If we denote by $a(\theta)$ the function defined by

$$
a(\theta) \;=\; 2a_1 \cos\theta + 2\tilde{a}_1 \sin\theta,
$$

we can use the operator T to regroup equations (5.44) and (5.45) in the following boundary condition

$$\frac{\partial\varphi}{\partial n} + T\varphi = a \quad \text{on } \Gamma_R.$$

Finally we have proved that the restriction of φ to B_R satisfies ($\frac{\partial\varphi}{\partial n}$ and φ are continuous across Γ_R)

$$(5.46)\qquad \left\{\begin{array}{l} \varphi \in V(B_R) = \{\varphi \in H^1(B_R) \ / \ \int_{\Gamma_R} \varphi \, d\sigma = 0\} \\ \mathrm{div}(\varepsilon\nabla\varphi) = 0 \\ \dfrac{\partial\varphi}{\partial n} + T\varphi = a. \end{array}\right.$$

If we can prove that, for any $a \in H^{1/2}(\Gamma_R)$, problem (5.46) has a unique solution in $V(B_R)$, this will prove that the space $\mathbf{P}_\varepsilon$ is isomorphic to the space

$$(5.47)\qquad \mathbf{B} = \{a(\theta) = 2a_1\cos\theta + 2\tilde{a}_1\sin\theta, \ (a_1, \tilde{a}_1) \in \mathbb{R}^2\}.$$

Indeed $\varphi\mid_{B_R}$ (and thus $\varphi\mid_{\Gamma_R}$) will be completely defined by the solution of (5.46) and then $\varphi\mid_{\mathbb{R}^2-B_R}$ completely determined by the formulas

$$(5.48)\qquad \left\{\begin{array}{l} \varphi(r) = \sum_{n=1}^{+\infty} \varphi_n(r)\cos(n\theta) + \tilde{\varphi}_n(r)\sin(n\theta) \\ \varphi_n(r) = \varphi_n(R)(\frac{r}{R})^{-n}, \quad \tilde{\varphi}_n(r) = \tilde{\varphi}_n(R)(\frac{r}{R})^{-n} \quad \text{for } n \ge 2 \\ \varphi_1(r) = \varphi_1(R)\frac{R}{r} + a_1 R(\frac{r}{R} - \frac{R}{r}), \quad \tilde{\varphi}_1(r) = \tilde{\varphi}_1(R)\frac{R}{r} + \tilde{a}_1 R(\frac{r}{R} - \frac{R}{r}). \end{array}\right.$$

From (5.46) and (5.48) one easily checks that $\nabla\varphi \in \mathbf{P}_\varepsilon$. Now for solving (5.46) we use the equivalent variational formulation

$$(5.49)\qquad \left\{\begin{array}{l} \text{Find } \varphi \in V(B_R) \text{ such that} \\ \int_{B_R} \varepsilon\nabla\varphi\nabla\psi \, dx + \varepsilon_\infty b(\varphi,\psi) = \varepsilon_\infty < a, \psi >_{\Gamma_R} \qquad \forall\psi \in V(B_R), \end{array}\right.$$

where $< .,. >_{\Gamma_R}$ denotes the duality $H^{\frac{1}{2}}(\Gamma_R) - H^{-\frac{1}{2}}(\Gamma_R)$ and the bilinear form $b(\varphi,\psi)$ is defined by

$$(5.50)\qquad b(\varphi,\psi) = \pi \sum_{n=1}^{+\infty} n\{\varphi_n(R)\overline{\psi_n}(R) + \tilde{\varphi}_n(R)\overline{\tilde{\psi}_n}(R)\}.$$

As $b(\varphi,\varphi) \ge 0$ and as in the space $V(B_R)$ we have a Poincaré 's inequality

$$(5.51)\qquad \int_{B_R} |\varphi|^2 \, dx \le C(R)\int_{B_R} |\nabla\varphi|^2 \, dx \quad \forall\varphi \in V(B_R),$$

it is clear that problem (5.49) is coercive in $V(B_R)$ and thus uniquely solvable by Lax-Milgram's lemma. The dimension of $\mathbf{B}$ being equal to 2, it is clear that $\mathbf{P}_\varepsilon$ is 2-dimensional and that a basis of $\mathbf{P}_\varepsilon$ can be obtained by taking successively ($a_1 = 1, \tilde{a}_1 = 0$) and ($a_1 = 0, \tilde{a}_1 = 1$) which gives the functions φ^1 and φ^2 defined in the theorem.

Remark 5.4 • *First, it is easy to show that $\mathbf{P}_\varepsilon \cap L^2(\Omega)^2$ is reduced to zero. Indeed $\boldsymbol{u} \in \mathbf{P}_\varepsilon$ implies that $\boldsymbol{u} = \nabla\varphi = a_1 \nabla\varphi^1 + \tilde{a}_1 \nabla\varphi^2$ where a_1 and $\tilde{a}_1$ are related to the first Fourier coefficients of φ^1 and φ^2 (see (5.34) for the definition) by the expressions*

$$\begin{cases} \varphi_1(r) = a_1 r + b_1 r^{-1} \\ \tilde{\varphi}_1(r) = \tilde{a}_1 r + \tilde{b}_1 r^{-1}. \end{cases}$$

But the fact that $\nabla\varphi \in L^2(\Omega)^2$ implies that $a_1 = \tilde{a}_1 = 0$. This yields $\boldsymbol{u} = 0$.

- *When ε is constant, it is not difficult to see that $\mathbf{P}_\varepsilon$ is the space of constant vector fields. An immediate way to see it, is to notice that constant vector fields belong to $\mathbf{P}_\varepsilon$ and to use the fact that* $\dim \mathbf{P}_\varepsilon = 2$. *One can also solve problems (5.31) explicitly. One obtains $\varphi^1 = x_1$ and $\varphi^2(x) = x_2$ which gives $\boldsymbol{u}^1 = (1,0)$ and $\boldsymbol{u}^2 = (0,1)$.*

Of course the equivalent of Lemma 5.3 holds for the space

$$\mathbf{P}_\mu = \{\boldsymbol{u} \in H_\rho(\mathrm{rot}, \mathrm{div}_\mu\,; \mathbb{R}^2) \,/\, \mathrm{rot}\boldsymbol{u} = \mathrm{div}(\mu\boldsymbol{u}) = 0\}\,. \tag{5.52}$$

Let us give two important properties of functions of spaces $\mathbf{P}_\varepsilon$ and $\mathbf{P}_\mu$ that we shall use in the sequel.

(i) A nonzero function $\boldsymbol{u}$ in the space $\mathbf{P}_\varepsilon$ (or $\mathbf{P}_\mu$) cannot be identically 0 in some open ball of $\mathbb{R}^2$.

(ii) The restrictions of u to $\mathbb{R}^2\backslash B_R$ belongs to the space $L^\infty(\mathbb{R}^2\backslash B_R)$, if $\boldsymbol{u}$ belongs to $\mathbf{P}_\varepsilon$ (or $\mathbf{P}_\mu$). The same holds for all its derivatives $D^\alpha \boldsymbol{u}$.

Property (i) is a consequence of a unique continuation theorem (see Theorem 1.2). Property (ii) comes from the fact that if $\boldsymbol{u}$ belongs to $\mathbf{P}_\varepsilon$ then $\boldsymbol{u} \in C^\infty(\overline{\mathbb{R}^2\backslash B_R})$ and is bounded at infinity. Indeed the C^∞-regularity of $\boldsymbol{u} = \nabla\varphi$ is a consequence of the one of φ, that one obtains by elliptic regularity. Moreover φ increases at most linearly at infinity while all its derivatives tend to 0 (see formulas (5.48)). It is then easy to prove (ii). We omit the details.

We can now give the
Proof of Theorem 5.4
Let $(\boldsymbol{w}_1, \boldsymbol{w}_2)$ the two functions of $L^2(\mathbb{R}^2)^2$ defined by

$$\begin{cases} \boldsymbol{w}_j(x) = \boldsymbol{u}^j(x) & \text{if} \quad |x| \leq R \\ \boldsymbol{w}_j(x) = 0 & \text{if} \quad |x| > R \end{cases} \tag{5.53}$$

where $\{\boldsymbol{u}^1, \boldsymbol{u}^2\}$ is a basis of $\mathbf{P}_\varepsilon$ (see Lemma 5.3). For each $j = 1, 2$, by Riesz' theorem, there exists a unique element $\tilde{w}_j$ in $H_\varepsilon(\beta)$ such that

$$\forall v \in H_\beta(\varepsilon) \qquad (\tilde{w}_j, v)_\varepsilon = \int_{\mathbb{R}^2} \boldsymbol{w}_j v \, dx.$$

Let F be the 2D-space generated by $\tilde{w}_1$ and $\tilde{w}_2$. By constructión we have

$$F^\perp = \{u \in H_\varepsilon(\beta) / \int_{\mathbb{R}^2} u\boldsymbol{w}_j \, dx = 0 \,; j = 1, 2\}\,. \tag{5.54}$$

Now let us recall that

$$c_\varepsilon(\beta\ ;u,u)\ =\ \beta^2\int_{B_R}\varepsilon(c^2-c_\infty^2)\mid \boldsymbol{u}\mid^2\ dx-2\beta\int_{B_R}\varepsilon(c^2-c_\infty^2)\nabla u_3\cdot\boldsymbol{u}\ dx\ .$$

Using Young's inequality as in the proof of Theorem 5.3, we obtain that ($\eta > 0$ can depend on x)

$$c_\varepsilon(\beta\ ;u,u)\ \geq\ \beta^2\int_{B_R}\varepsilon(c^2-c_\infty^2)\left[1-\frac{\varepsilon(c^2-c_\infty^2)}{\eta}\right]\mid \boldsymbol{u}\mid^2\ dx-\int_{\mathbb{R}^2}\eta\mid\nabla u_3\mid^2\ dx\ .$$

Choosing $\eta = \varepsilon c^2$, we get, for any u in $V_\varepsilon(\beta)$

$$(5.55)\quad \left|\begin{array}{l} p_\varepsilon(\beta\ ;u,u)+c_\varepsilon(\beta\ ;u,u)\geq\ \beta^2c_\infty^2\displaystyle\int_{\mathbb{R}^2}\varepsilon\frac{(c^2-c_\infty^2)}{c^2}\mid \boldsymbol{u}\mid^2\ dx\\ \qquad\qquad+\ \beta^2c_\infty^2\displaystyle\int_{\mathbb{R}^2}\varepsilon\mid u_3\mid^2\ dx+\int_{\mathbb{R}^2}\varepsilon c^2\mid \mathrm{rot}\boldsymbol{u}\mid^2\ dx\ .\end{array}\right.$$

To conclude, we shall admit for the moment Lemma 5.4, which allows us to obtain the following estimation (since one has $\beta u_3 = \dfrac{1}{\varepsilon}\mathrm{div}(\varepsilon u)$)

$$\forall u\in F^\perp\cap V_\varepsilon(\beta)\quad \beta^2c_\infty^2\int_{\mathbb{R}^2}\varepsilon\mid u_3\mid^2\ dx+\int_{\mathbb{R}^2}\varepsilon c^2\mid \mathrm{rot}\boldsymbol{u}\mid^2\ dx\ \geq\ C(R)\int_{B_R}\varepsilon\mid \boldsymbol{u}\mid^2\ dx.$$

Therefore, using (2.2) and (5.55), we get

$$(5.56)\quad \left|\begin{array}{l}\forall u\in F^\perp\quad a_\varepsilon(\beta\ ;u,u)-c_\infty^2\beta^2\mid u\mid_\varepsilon^2\ \geq\\ \qquad\left\{C(R)-\beta^2c_\infty^2\parallel\dfrac{c^2-c_\infty^2}{c^2}\parallel_{L^\infty}\right\}\displaystyle\int_{B_R}\varepsilon\mid \boldsymbol{u}\mid^2\ dx\ .\end{array}\right.$$

In particular, we have

$$\beta\leq\left(\frac{C(R)}{c_\infty^2\parallel\frac{c^2-c_\infty^2}{c^2}\parallel_{L^\infty}}\right)^{\frac{1}{2}}\quad\Longrightarrow\quad a_\varepsilon(\beta\ ;u,u)\ \geq\beta^2c_\infty^2\mid u\mid_\varepsilon^2\quad\forall u\in F^\perp\ .$$

By the Min-Max principle this means that $s_3(\beta)\ =\ c_\infty^2\beta^2$ and thus that the third lower threshold β_3^0 satisfies

$$(5.57)\qquad \beta_3^0\ \geq\ \left(\frac{C(R)}{c_\infty^2\parallel c^2-c_\infty^2\parallel_{L^\infty}}\right)^{\frac{1}{2}}\ >\ 0.$$

We show now

Lemma 5.4 *There exists a constant $C(R) > 0$ such that*

$$(5.58)\quad \left\{\begin{array}{l}\forall u=(\boldsymbol{u},u_3)\in V_\varepsilon(\beta)\cap F^\perp\\ C(R)\displaystyle\int_{B_R}\varepsilon\mid \boldsymbol{u}\mid^2\ dx\ \leq\ \left\{\int_{\mathbb{R}^2}\left(\varepsilon c^2\mid rot\boldsymbol{u}\mid^2+\frac{c_\infty^2}{\varepsilon}\mid div\varepsilon\boldsymbol{u}\mid^2\right)dx\right\}\end{array}\right.$$

Proof Assume (5.58) is false. Then there would exist a sequence u^n in the space $F^{\perp} \cap V_{\varepsilon}(\beta)$ such that

$$\begin{cases} \displaystyle\int_{B_R} \varepsilon \mid \boldsymbol{u}^n \mid^2 \, dx = 1 \\ \displaystyle\int_{\mathbf{R}^2} \left\{ \varepsilon c^2 \mid \mathrm{rot}\boldsymbol{u}^n \mid^2 + \frac{c_\infty^2}{\varepsilon} \mid \mathrm{div}(\varepsilon \boldsymbol{u}_n) \mid^2 \right\} \, dx \le \frac{1}{n}. \end{cases}$$

Using Proposition (5.1), we can extract from u^n a sequence such that

$$\left| \begin{array}{lll} \boldsymbol{u}^n \rightharpoonup \boldsymbol{u} & \text{weakly in} & H_\rho(\mathrm{rot}, \mathrm{div}_\varepsilon \, ; \mathbf{R}^2) \\ \boldsymbol{u}^n \to \boldsymbol{u} & \text{strongly in} & L^2(B_R)^2 \\ \mathrm{rot}\boldsymbol{u}^n \to 0 & \text{strongly in} & L^2(\mathbf{R}^2) \\ \mathrm{div}(\varepsilon \boldsymbol{u}^n) \to 0 & \text{strongly in} & L^2(\mathbf{R}^2) \end{array} \right.$$

The limit field $\boldsymbol{u}$ then satisfies

$$(5.59) \qquad \left| \begin{array}{l} \boldsymbol{u} \in H_\rho(\mathrm{rot}, \mathrm{div}_\varepsilon \, ; \mathbf{R}^2) \quad \mathrm{rot}\boldsymbol{u} = \mathrm{div}(\varepsilon \boldsymbol{u}) = 0 \\ \displaystyle\int_{\mathbf{R}^2} \boldsymbol{u} \cdot \boldsymbol{w}_j \, dx = 0 \quad j = 1,2 \quad (\text{since } \boldsymbol{u}^n \in F^{\perp}) \\ \displaystyle\int_{B_R} \varepsilon \mid \boldsymbol{u} \mid^2 \, dx = 1. \end{array} \right.$$

From the two first equalities of (5.59) we deduce that $\boldsymbol{u}$ belongs to $\mathbf{P}_\varepsilon$ and thus that there exists $(\alpha_1, \alpha_2) \in \mathbf{R}^2$ such that

$$(5.60) \qquad \boldsymbol{u} = \alpha_1 \boldsymbol{u}^1 + \alpha_2 \boldsymbol{u}^2 \, .$$

The orthogonality conditions $\int_{\mathbf{R}^2} \varepsilon \boldsymbol{u} \boldsymbol{w}_j \, dx = 0$ $(j = 1,2)$ thus lead to the homogeneous system

$$(5.61) \qquad \begin{cases} \left(\displaystyle\int_{B_R} (\boldsymbol{u}^1)^2 \, dx \right) \alpha_1 + \left(\displaystyle\int_{B_R} \boldsymbol{u}^1 \boldsymbol{u}^2 \, dx \right) \alpha_2 = 0 \\ \left(\displaystyle\int_{B_R} \boldsymbol{u}^1 \boldsymbol{u}^2 \, dx \right) \alpha_1 + \left(\displaystyle\int_{B_R} (\boldsymbol{u}^2)^2 \, dx \right) \alpha_2 = 0 \end{cases}$$

whose unique solution is $\alpha_1 = \alpha_2 = 0$ since $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ are linearly independent as elements of $L^2(B_R)$. Thus by (5.60), $\boldsymbol{u} \equiv 0$ which contradicts the last equality of (5.59). This completes the proof.

5.5 Asymptotic behaviour of the thresholds. Finiteness of $N(\beta)$

Our objective in this paragraph is to prove that

$$(5.62) \qquad \lim_{m \to +\infty} \beta_m^0 = +\infty$$

which will imply the finiteness of $N(\beta)$ for each β. We could use the fact that we have a comparison result by Theorem 5.2 and compare our medium with a simpler one for which the calculations can be made by hand. This is the case if we choose

$$\begin{cases} \varepsilon(x) = \varepsilon^+ \quad \text{and} \quad \mu(x) = \mu^+ \quad \text{if} \quad |x| \leq R \\ \varepsilon(x) = \varepsilon_\infty \quad \text{and} \quad \mu(x) = \mu_\infty \quad \text{if} \quad |x| > R. \end{cases}$$

For such a medium the exact relation dispersion exists. One could derive from this exact dispersion relation an equation for the thresholds and study this equation. Such calculations have already been done when μ is constant by R. Djellouli in [13]. The resulting threshold equation is very complicated and the results presented in [13] cannot be considered as mathematically proven and anyway do not concern the behaviour of the thresholds. Such an approach appearing as complicated and painful, we have preferred to follow the more elegant one, and probably more general, developed by A. Bamberger and A.S. Bonnet in [2]. We first consider the particular case

(5.63) $$c(x) \leq c_\infty \quad a.e. x \in \mathbb{R}^2.$$

In that case we know (cf. Theorem 5.1) that

$$\beta_m^* = \beta_m^0 \quad \forall m \geq 1.$$

The intuitive idea of the proof consists in saying that if the sequence β_m^* accumulates to some finite value $\beta^* \neq 0$ then β^* must be solution of the threshold equation and of its derivative with respect to β. One then checks that these two equations are not compatible as soon as (5.63) is satisfied. The complete proof is slightly more complicated. First, for purely technical reasons, we shall make for the moment the following assumption:

(5.64) $$\exists \quad D_\gamma = \{x / \mid x - x_0 \mid < \gamma\} \quad / \quad c(x) < c_\infty \quad \text{a.e. } x \in D_\gamma.$$

Then we prove a first lemma.

Lemma 5.5 *Assume that (5.63) and (5.64) hold. Let β_p be a sequence of solutions of the threshold equation satisfying*

$$\begin{cases} \bullet \quad p \neq q \quad \Longrightarrow \quad \beta_p \neq \beta_q \\ \bullet \quad \lim_{p \to +\infty} \beta_p = \beta^*, \end{cases}$$

then $\beta^* = 0$.

Proof By definition of the threshold equation (5.17) one can construct a sequence u_p in $V_{\rho,\varepsilon}(\beta_p)$ such that

(5.65) $$\begin{cases} p_\varepsilon(\beta_p \, ; u_p, v) + c_\varepsilon(\beta_p \, ; u_p, v) = 0 \quad \forall v \in V_{\rho,\varepsilon}(\beta_p) \\ \displaystyle\int_{B_R} \varepsilon \mid \boldsymbol{u}_p \mid^2 dx = 1 \end{cases}$$

(we have used the fact that $\boldsymbol{u}_p$ is defined up to a multiplicative constant and that it cannot vanish identically in the ball B_R).

1) Using the notation of the proof of Theorem 5.3, the function $\tilde{u}_p = (\boldsymbol{u}_p, \tilde{u}_{3,p} = \beta_p u_{3,p})$ is solution of

(5.66) $$\tilde{p}_\varepsilon(\beta_p \, ; \tilde{u}_p, \tilde{v}) + \tilde{c}_\varepsilon(\beta_p \, ; \tilde{u}_p, \tilde{v}) = 0 \quad \forall \tilde{v} \in V_{\rho,\varepsilon}(1)$$

where we have set, for any $\beta > 0$,

$$(5.67)\quad \left| \begin{array}{lll} \tilde{p}_\varepsilon(\beta\ ;\tilde{u},\tilde{v}) & = & \displaystyle\int_{\mathbb{R}^2} \varepsilon c^2(\mathrm{rot}\boldsymbol{u}\,\mathrm{rot}\boldsymbol{v} + \frac{1}{\beta^2}\nabla\tilde{u}_3\cdot\nabla\tilde{v}_3)\ dx + c_\infty^2 \int_{\mathbb{R}^2} \varepsilon\tilde{u}_3\tilde{v}_3\ dx \\ \tilde{c}_\varepsilon(\beta\ ;\tilde{u},\tilde{v}) & = & \displaystyle\beta^2\int_{\mathbb{R}^2} \varepsilon(c^2-c_\infty^2)\boldsymbol{u}\cdot\boldsymbol{v}\ dx - \int_{\mathbb{R}^2}\varepsilon(c^2-c_\infty^2)[\nabla\tilde{u}_3\cdot\boldsymbol{v}+\nabla\tilde{v}_3\cdot\boldsymbol{u}]\ dx\ . \end{array}\right.$$

Assume that $\beta_p \to \beta^* \neq 0$. Then reasoning as in Theorem 5.3, we deduce that one can extract from $\tilde{u}_p$ a subsequence, still denote $\tilde{u}_p$ such that

$$\left\{ \begin{array}{lll} \tilde{u}_p \rightharpoonup \tilde{u} & \text{weakly in} & V_{\rho,\varepsilon}(1) \\ \tilde{u}_p \to \tilde{u} & \text{strongly in} & L^2(B_R)^3 \end{array}\right.$$

and we have at the limit

$$(5.68)\qquad \tilde{p}_\varepsilon(\beta^*\ ;\tilde{u},\tilde{v}) + \tilde{c}_\varepsilon(\beta^*\ ;\tilde{u},\tilde{v}) = 0 \qquad \forall\tilde{v}\in V_{\rho,\varepsilon}(1)\ .$$

2) Now let us take $\tilde{v} = \tilde{u}_q$ in (5.66) we obtain

$$(5.69)\qquad \tilde{p}_\varepsilon(\beta_p\ ;\tilde{u}_p,\tilde{u}_q) + \tilde{c}_\varepsilon(\beta_p\ ;\tilde{u}_p,\tilde{u}_q) = 0$$

and inverting the rôle of p and q we have

$$(5.70)\qquad \tilde{p}_\varepsilon(\beta_q\ ;\tilde{u}_p,\tilde{u}_q) + \tilde{c}_\varepsilon(\beta_q\ ;\tilde{u}_p,\tilde{u}_q) = 0\ .$$

Taking the difference between (5.69) and (5.70) we have

$$\left(\frac{1}{\beta_q^2}-\frac{1}{\beta_p^2}\right)\int_{\mathbb{R}^2}\varepsilon c^2\nabla\tilde{u}_{3,q}\cdot\nabla\tilde{u}_{3,p}\ dx\ +\ (\beta_q^2-\beta_p^2)\int_{\mathbb{R}^2}\varepsilon(c^2-c_\infty^2)\boldsymbol{u}_p\cdot\boldsymbol{u}_q\ dx = 0,$$

that is to say, after division by $\beta_p^2 - \beta_q^2 \neq 0$

$$(5.71)\qquad \frac{1}{\beta_p^2\beta_q^2}\int_{\mathbb{R}^2}\varepsilon c^2\nabla\tilde{u}_{3,q}\cdot\nabla\tilde{u}_{3,p}\ dx - \int_{\mathbb{R}^2}\varepsilon(c^2-c_\infty^2)\boldsymbol{u}_p\cdot\boldsymbol{u}_q\ dx = 0.$$

This equality being satisfied for any p and q we can pass to the limit ($p\to+\infty$ first, then $q\to+\infty$) to obtain

$$(5.72)\qquad \frac{1}{(\beta^*)^4}\int_{\mathbb{R}^2}\varepsilon c^2\mid\nabla\tilde{u}_3\mid^2\ dx - \int_{\mathbb{R}^2}\varepsilon(c^2-c_\infty^2)\mid\boldsymbol{u}\mid^2\ dx = 0.$$

3) Because of assumption (5.63) this implies

$$(5.73)\qquad \tilde{u}_3\ =\ 0 \text{ and } \int_{\mathbb{R}^2}\varepsilon(c^2-c_\infty^2)\mid\boldsymbol{u}\mid^2\ dx\ =\ 0.$$

Then, coming back to (5.68) with $\tilde{v} = \tilde{u}$ we obtain $\mathrm{rot}\boldsymbol{u} = 0$. This means, together with $\tilde{u}_3 = 0$, that $\tilde{\boldsymbol{u}}$ belongs to $\mathbf{P}_\varepsilon$. From (5.63),(5.64) and (5.73) we also deduce that $\tilde{u}$ is identically 0 in the disk D_γ. Finally one has $\tilde{u} = 0$ everywhere by unique continuation theorem (cf. Remark 5.4). But this contradicts the fact that $\displaystyle\int_{B_R}\varepsilon\mid\boldsymbol{u}\mid^2\ dx = 1$ which stems up from the local strong convergence of the sequence $\boldsymbol{u}_p$.

We now prove (5.62) when (5.63) and (5.64) hold.

Lemma 5.6 *Under the assumptions of Lemma 5.5, the sequence β_m^0 tends to $+\infty$ when m tends to $+\infty$*

Proof The sequence β_m^* is increasing. Assume that $\beta_m^* \to \beta^* < +\infty$. As $\beta_3^* \geq \beta_3^0 > 0$ by Theorem (5.4), we necessarily have $\beta^* > 0$. Now we remark that any $\beta > \beta^*$ is solution of the threshold equation (5.17). Indeed since $\beta > \beta^*$ then $\beta > \beta_m^*$. Therefore, by definition of β_m^*, the operator $A_\varepsilon(\beta)$ has an infinity of eigenvalues $\omega_m(\beta)^2 < c_\infty^2\beta^2$ which satisfies by Min-Max principle,

$$\lim_{m\to+\infty} \omega_m(\beta)^2 = c_\infty^2\beta^2 \ .$$

We can then repeat identically the proof of Theorem 5.3 (except that we have a fixed β here instead of a converging sequence β_p) to conclude that β is a solution of the threshold equation. Therefore we can construct a strictly decreasing sequence β_p satisfying the assumptions of Lemma 5.5 and the fact that $\beta^* > 0$ is in contradiction with this lemma. This completes the proof.

We now prove (5.62) for the general case.

Theorem 5.5 *For any medium the sequence β_m^0 goes to $+\infty$ when $m \to +\infty$. Consequently the number $N(\beta)$ of guided modes is finite for any $\beta > 0$.*

Proof We use a comparison technique. Let us introduce $\tilde{\varepsilon}(x)$ and $\tilde{\mu}(x)$ defined by

$$\begin{cases} \tilde{\varepsilon}(x) = (\varepsilon^+ + 1) \quad \text{and} \quad \tilde{\mu}(x) = (\mu^+ + 1) \quad \text{if } \mid x \mid \leq R \\ \\ \tilde{\varepsilon}(x) = \varepsilon_\infty \qquad\quad \text{and} \quad \tilde{\mu}(x) = \mu_\infty \qquad\quad \text{if } \mid x \mid > R. \end{cases}$$

By construction we have

$$\tilde{\varepsilon}(x) \geq \varepsilon(x), \quad \tilde{\mu}(x) \geq \mu(x) \quad \text{and} \quad \tilde{c}(x) < c_\infty \quad \text{a.e. } x \in B_R.$$

We can apply Lemma 5.6 to the medium $(\tilde{\varepsilon}, \tilde{\mu})$ to assert that, with obvious notation,.

$$\lim_{m\to+\infty} \tilde{\beta}_m^0 \ = \ +\infty \ .$$

But by Theorem 5.2, $\beta_m^0 \geq \tilde{\beta}_m^0$, therefore

$$\lim_{m\to+\infty} \beta_m^0 = +\infty \ .$$

5.6 Existence and nonexistence results for small values of β

In this section we investigate the possibility of existence of guided modes for small β or equivalently the fact that a threshold can be equal to 0. From Section 5.4, we know that at most two modes can propagate for small β but we don't know if the two first thresholds β_1^0 and β_2^0 are equal to 0 or not. Our first result gives a necessary condition for such a situation to occur. Let us set

$$\begin{aligned} B(\mathbf{P}_\varepsilon) \ &= \ \left\{ \boldsymbol{u} \in \mathbf{P}_\varepsilon \ / \ \int_{B_R} \varepsilon|\boldsymbol{u}|^2 \, dx = 1. \right\} \\ B(\mathbf{P}_\mu) \ &= \ \left\{ \boldsymbol{u} \in \mathbf{P}_\mu \ / \ \int_{B_R} \mu|\boldsymbol{u}|^2 \, dx = 1. \right\}. \end{aligned}$$

Let us first prove the following lemma.

Lemma 5.7 *If* $\beta_1^0 = 0$, *then*

$$\min_{\boldsymbol{u}\in B(\mathbf{P}_\epsilon)} \int_{\mathbb{R}^2} \varepsilon \left(\frac{1}{c_\infty^2} - \frac{1}{c^2}\right) | \boldsymbol{u} |^2 \ dx \leq 0.$$

Proof If $\beta_1^0 = 0$, we can construct a strictly decreasing sequence $\beta_p \searrow 0$ such that, for any $p \geq 1$, there exists $(u_p, \omega_p^2) \in V_\epsilon(\beta_p) \times \mathbb{R}_*^+$ such that

$$(5.74) \qquad \left\{ A_\epsilon(\beta_p)u_p = \omega_p^2 u_p, \quad \omega_p^2 < c_\infty^2 \beta_p^2 \quad \text{and} \quad \int_{B_R} \varepsilon \mid u_p \mid^2 \ dx = 1. \right.$$

1) Defining $\tilde{u}_p = (\boldsymbol{u}_p, \tilde{u}_{3,p} = \beta_p u_{3,p})$ and proceeding exactly as in the proof of Theorem 5.3, we can write inequality (5.24) for $\tilde{u}_p$ from which we deduce the following convergences, up to the extraction of a subsequence (the difference with the sequence $\boldsymbol{u}_p$ in Theorem 5.3 comes from the fact that here β_p tends to 0):

$$\left\{ \begin{array}{lll} \boldsymbol{u}_p \rightharpoonup \boldsymbol{u} & \text{weakly in} & H(\text{rot}, \text{div}_\epsilon \ ; \mathbb{R}^2) \\ \boldsymbol{u}_p \to \boldsymbol{u} & \text{strongly in} & L^2(B_R)^2 \\ \text{rot}\boldsymbol{u}_p \to 0 & \text{strongly in} & L^2(\mathbb{R}^2) \\ \tilde{u}_{3,p} \to 0 & \text{strongly in} & H^1(\mathbb{R}^2), \end{array} \right.$$

with moreover at the limit: $\boldsymbol{u} \in B(\mathbf{P}_\epsilon)$.

2) We also have, as u_p is an eigenfunction and $\omega_p^2 < c_\infty^2 \beta_p^2$,

$$(5.75) \qquad p_\epsilon(\beta_p \ ; u_p, u_p) + c_\epsilon(\beta_p \ ; u_p, u_p) < 0 \ .$$

But using inequality (5.55), that we write in terms of $\tilde{u}_p$ instead of u_p, we deduce from (5.75) that we also have

$$\beta_p^2 c_\infty^4 \int_{B_R} \varepsilon \left(\frac{1}{c_\infty^2} - \frac{1}{c^2}\right) | \boldsymbol{u}_p |^2 \ dx + c_\infty^2 \int_{\mathbb{R}^2} \varepsilon \mid \tilde{u}_{3,p} \mid^2 \ dx + \int_{\mathbb{R}^2} \varepsilon c^2 \text{rot} \mid \boldsymbol{u}_p \mid^2 \ dx \leq 0 \ .$$

In particular, the term $\int_{B_R} \varepsilon \left(\frac{1}{c_\infty^2} - \frac{1}{c^2}\right) | \boldsymbol{u}_p |^2 \ dx$ is negative for any p, so that, passing to the limit when $p \to +\infty$, we obtain

$$\int_{B_R} \varepsilon \left(\frac{1}{c_\infty^2} - \frac{1}{c^2}\right) | \boldsymbol{u} |^2 \ dx \leq 0,$$

which leads to the desired result.

As a minor modification of Lemma 5.7 we have the

Lemma 5.8 *If* $\beta_2^0 = 0$, *then*

$$\max_{\boldsymbol{u}\in B(\mathbf{P}_\epsilon)} \int_{B_R} \varepsilon \left(\frac{1}{c_\infty^2} - \frac{1}{c^2}\right) | \boldsymbol{u} |^2 \ dx \leq 0.$$

Proof Proceeding as in the proof of Lemma 5.7 we can work with 2 sequences $\boldsymbol{u}_p^{(1)}$ and $\boldsymbol{u}_p^{(2)}$ satisfying the orthogonality condition

$$\int_{B_R} \varepsilon \boldsymbol{u}_p^{(1)}.\boldsymbol{u}_p^{(2)} \, dx = 0.$$

At the limit we construct two functions $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ of $B(\mathbf{P}_\varepsilon)$ satisfying

$$\left| \begin{array}{l} \displaystyle\int_{B_R} \varepsilon \boldsymbol{u}^{(1)}.\boldsymbol{u}^{(2)} \, dx = 0 \\ \displaystyle\int_{B_R} \varepsilon \left(\frac{1}{c_\infty^2} - \frac{1}{c^2}\right) | \boldsymbol{u}^{(1)} |^2 \; dx \leq 0 \text{ and } \int_{B_R} \varepsilon \left(\frac{1}{c_\infty^2} - \frac{1}{c^2}\right) | \boldsymbol{u}^{(2)} |^2 \; dx \leq 0. \end{array} \right.$$

The same process can be applied to the sequence

$$\boldsymbol{u}_p = \alpha_1 \boldsymbol{u}_p^{(1)} + \alpha_2 \boldsymbol{u}_p^{(2)}$$

for any (α_1, α_2) such that $\alpha_1^2 + \alpha_2^2 = 1$. At the limit we obtain

$$\int_{B_R} \varepsilon \left(\frac{1}{c_\infty^2} - \frac{1}{c^2}\right) | \alpha_1 \boldsymbol{u}^{(1)} + \alpha_2 \boldsymbol{u}^{(2)} |^2 \; dx \; \leq \; 0.$$

This leads to the result since $\dim \mathbf{P}_\varepsilon = 2$.

Playing with the dual formulation between $A_\varepsilon(\beta)$ and $A_\mu(\beta)$, we finally obtain the following result.

Theorem 5.6 *(i) If $\beta_1^0 = 0$, then*

$$\max\left(\min_{\boldsymbol{u} \in B(\mathbf{P}_\varepsilon)} \int_{B_R} \varepsilon \left(\frac{1}{c_\infty^2} - \frac{1}{c^2}\right) |\boldsymbol{u}|^2 dx, \min_{\boldsymbol{u} \in B(\mathbf{P}_\mu)} \int_{B_R} \mu \left(\frac{1}{c_\infty^2} - \frac{1}{c^2}\right) |\boldsymbol{u}|^2 dx \right) \leq 0 \tag{5.76}$$

(ii) If $\beta_2^0 = 0$, then

$$\max\left(\max_{\boldsymbol{u} \in B(\mathbf{P}_\varepsilon)} \int_{B_R} \varepsilon \left(\frac{1}{c_\infty^2} - \frac{1}{c^2}\right) |\boldsymbol{u}|^2 dx, \max_{\boldsymbol{u} \in B(\mathbf{P}_\mu)} \int_{B_R} \mu \left(\frac{1}{c_\infty^2} - \frac{1}{c^2}\right) |\boldsymbol{u}|^2 \; dx \right) \leq 0 \tag{5.77}$$

Comments:

- It is clear that the conditions expressed in points (i) and (ii) of Theorem 5.6 are symmetric with respect to ε and μ.
- Remarking that if we set $R\boldsymbol{u} = (-u_2, u_1)$ we have the equivalences:

$$\left\{ \begin{array}{lll} \boldsymbol{u} \in \mathbf{P}_\varepsilon & \Longleftrightarrow & \boldsymbol{v} = \varepsilon R \boldsymbol{u} \in \mathbf{P}_{\frac{1}{\varepsilon}} \\ \boldsymbol{u} \in \mathbf{P}_\mu & \Longleftrightarrow & \boldsymbol{v} = \mu R \boldsymbol{u} \in \mathbf{P}_{\frac{1}{\mu}} \end{array} \right.$$

we can give another form of conditions (5.76) and (5.77):

$$\left| \begin{array}{l} (5.76) \iff \max\left(\displaystyle\min_{\boldsymbol{u} \in B(\mathbf{P}_{\frac{1}{\mu}})} \int_{B_R} (c^2 - c_\infty^2) \, | \boldsymbol{u} |^2 \, \varepsilon dx, \min_{\boldsymbol{u} \in B(\mathbf{P}_{\frac{1}{\varepsilon}})} \int_{B_R} (c^2 - c_\infty^2) \, | \boldsymbol{u} |^2 \, \mu \; dx \right) \leq 0 \\ (5.77) \iff \max\left(\displaystyle\max_{\boldsymbol{u} \in B(\mathbf{P}_{\frac{1}{\mu}})} \int_{B_R} (c^2 - c_\infty^2) \, | \boldsymbol{u} |^2 \, \varepsilon dx, \max_{\boldsymbol{u} \in B(\mathbf{P}_{\frac{1}{\varepsilon}})} \int_{B_R} (c^2 - c_\infty^2) \, | \boldsymbol{u} |^2 \, \mu \; dx \right) \leq 0 \, . \end{array} \right.$$

We are now looking at a kind of reciprocal to Theorem 5.6 which consists in looking for sufficient conditions on ε and μ in order to ensure the existence of guided modes for any value of β.

Theorem 5.7 *(i) If we suppose that*

$$\min\left(\min_{\boldsymbol{u}\in B(\mathbf{P}_\varepsilon)}\int_{B_R}\varepsilon(c^2-c_\infty^2)|\boldsymbol{u}|^2\,dx,\ \min_{\boldsymbol{u}\in B(\mathbf{P}_\mu)}\int_{B_R}\mu(c^2-c_\infty^2)|\boldsymbol{u}|^2\,dx\right)\le 0 \tag{5.78}$$

the first upper threshold β_1^* is equal to 0 which means that $A_\varepsilon(\beta)$ admits at least one eigenvalue strictly smaller than $c_\infty^2\beta^2$ for any $\beta>0$.

(ii) If we suppose that

$$\min\left(\max_{\boldsymbol{u}\in B(\mathbf{P}_\varepsilon)}\int_{B_R}\varepsilon(c^2-c_\infty^2)|\boldsymbol{u}|^2\,dx,\ \max_{\boldsymbol{u}\in B(\mathbf{P}_\mu)}\int_{B_R}\mu(c^2-c_\infty^2)|\boldsymbol{u}|^2\,dx\right)\le 0 \tag{5.79}$$

then $\beta_1^*=\beta_2^*=0$ which means that $A_\varepsilon(\beta)$ admits at least two eigenvalues strictly smaller than $c_\infty^2\beta^2$ for any $\beta>0$.

Proof **(ia)** We first assume that the inequality (5.78) is strict and that

$$\min_{\boldsymbol{u}\in B(\mathbf{P}_\varepsilon)}\int_{B_R}\varepsilon(c^2-c_\infty^2)\mid\boldsymbol{u}\mid^2\,dx\le\min_{\boldsymbol{u}\in B(\mathbf{P}_\mu)}\int_{B_R}\mu(c^2-c_\infty^2)\mid\boldsymbol{u}\mid^2\,dx.$$

If not we simply invert the rôles of ε and μ in our proof. We can thus choose $\boldsymbol{u}$ in $\mathbf{P}_\varepsilon$ such that

$$\int_{B_R}\varepsilon(c^2-c_\infty^2)\mid\boldsymbol{u}\mid^2\,dx<0. \tag{5.80}$$

Our method is closed to the one of [2] but we have to deal with technical difficulties. The idea is that we would like to take $(\boldsymbol{u},0)$ as a test function for applying the Min-Max principle. This is not possible since $(\boldsymbol{u},0)$ does not belong to $V_\varepsilon(\beta)$. This is why we use a truncation-regularization process. First consider $d>0$ and $M>R+d$ (M will tend to $+\infty$) and define the classical functions (cf. H. Picq's Thesis [24])

$$\left\{\begin{array}{ll} v_M(x)=1 & \text{if } |x|\le R+d\\[2mm] v_M(x)=\dfrac{Log(\frac{M}{|x|})}{Log(\frac{M}{(R+d)})} & \text{if } R+d\le|x|\le M\\[2mm] \text{supp } v_M\subset\{\mid x\mid\le M\}. & \end{array}\right. \tag{5.81}$$

For any M, $v_M\in H^1(\mathbf{R}^2)$ and moreover when $M\to+\infty$

$$\int_{\mathbf{R}^2}\mid\nabla v_M\mid^2\,dx\underset{M\to+\infty}{\sim}\frac{C}{Log M}. \tag{5.82}$$

We would like to take $v_M\boldsymbol{u}$ as the transverse field of our test function but $\operatorname{div}(\varepsilon v_M\boldsymbol{u})\notin H^1(\mathbf{R}^2)$ and that is why we need regularization. Introduce $\zeta(x)$ defined by

$$\zeta\in C_0^\infty(\mathbf{R}^2),\quad \text{supp }\zeta\subset\{x\ /\ \mid x\mid<d\}$$

and set

$$\psi_M=\zeta*v_M\quad\in\quad C^\infty(\mathbf{R}^2). \tag{5.83}$$

It is immediate from convolution properties that

$$\begin{cases} \psi_M(x) = 1 & \text{if} \quad |x| \leq R \\ \psi_M(x) = 0 & \text{if} \quad |x| \geq M + d \end{cases}$$

Moreover, we have

$$\left| \begin{array}{lcl} \dfrac{\partial \psi_M}{\partial x_i} = \zeta * \dfrac{\partial v_M}{\partial x_i} & \Longrightarrow & \| \dfrac{\partial \psi_M}{\partial x_i} \|_{L^2} \leq \| \zeta \|_{L^1} \| \dfrac{\partial v_M}{\partial x_i} \|_{L^2} \\ \dfrac{\partial^2 \psi_M}{\partial x_i \partial x_j} = \dfrac{\partial \zeta}{\partial x_i} * \dfrac{\partial v_M}{\partial x_j} & \Longrightarrow & \| \dfrac{\partial^2 \psi_M}{\partial x_i \partial x_j} \|_{L^2} \leq \| \dfrac{\partial \zeta}{\partial x_i} \|_{L^1} \| \dfrac{\partial v_M}{\partial x_j} \|_{L^2}, \end{array} \right.$$

so that for $(i,j) \in \{1,2\}^2$, we have

$$\lim_{M \to +\infty} \| \frac{\partial \psi_M}{\partial x_i} \|_{L^2} = \lim_{M \to +\infty} \| \frac{\partial^2 \psi_M}{\partial x_i \partial x_j} \|_{L^2} = 0. \tag{5.84}$$

We now define

$$\boldsymbol{u}_M = (\boldsymbol{u}_M, \boldsymbol{u}_{3,M}) = \left(\psi_M \boldsymbol{u}, \frac{1}{\varepsilon \beta} \operatorname{div}(\varepsilon \psi_M \boldsymbol{u}) \right) \in V_\varepsilon(\beta). \tag{5.85}$$

We can remark that, as $\operatorname{rot} \boldsymbol{u} = 0$

$$\operatorname{rot} \boldsymbol{u}_M = \frac{\partial \psi_M}{\partial x_1} \boldsymbol{u}_2 - \frac{\partial \psi_M}{\partial x_2} \boldsymbol{u}_1$$

so that, as $\nabla \psi_M = 0$ if $|x| \leq R$ and $\boldsymbol{u} \in L^\infty(\mathbf{R}^2 \backslash B_R)$ (cf. also Remark 5.4)

$$\| \operatorname{rot} \boldsymbol{u}_M \|_{L^2} \leq \| \boldsymbol{u} \|_{L^\infty(\mathbf{R}^2 - B_R)} \| \nabla \psi_M \|_{L^2} \to 0 \text{ when } M \to +\infty). \tag{5.86}$$

In the same way, as $\operatorname{div}(\varepsilon \boldsymbol{u}) = 0$, we have

$$\begin{cases} u_{3,M} &= \dfrac{1}{\beta} \nabla \psi_M \cdot \boldsymbol{u} \\ \dfrac{\partial u_{3,M}}{\partial x_i} &= \dfrac{1}{\beta} \left(\dfrac{\partial}{\partial x_i} (\nabla \psi_M) \cdot \boldsymbol{u} + \nabla \psi_M \cdot \dfrac{\partial \boldsymbol{u}}{\partial x_i} \right) \end{cases}$$

so that (since $\boldsymbol{u} \in W^{1,\infty}(\mathbf{R}^2 \backslash B_R)$), we obtain

$$\left| \begin{array}{lcl} \| u_{3,M} \|_{L^2} & \leq & \dfrac{1}{\beta} \| \boldsymbol{u} \|_{L^\infty(\mathbf{R}^2 \backslash B_R)} \| \nabla \psi_M \|_{L^2} \quad \to \quad 0 \quad (M \to +\infty) \\ \| \dfrac{\partial u_{3,M}}{\partial x_i} \|_{L^2} & \leq & \dfrac{1}{\beta} \| \dfrac{\partial}{\partial x_i} (\nabla \psi_M) \|_{L^2} \| \boldsymbol{u} \|_{L^\infty(\mathbf{R}^2 \backslash B_R)} + \| \nabla \psi_M \|_{L^2} \| \dfrac{\partial \boldsymbol{u}}{\partial x_i} \|_{L^\infty(\mathbf{R}^2 \backslash B_R)} \\ & & \to \ 0 \quad (M \to +\infty) . \end{array} \right. \tag{5.87}$$

Finally remark that as $\psi_M = 1$ in B_R,

$$\boldsymbol{u}_M = \boldsymbol{u}, \quad u_{3,M} = 0 \qquad \text{a.e. } x \in B_R. \tag{5.88}$$

Now, using (2.2), we write

$$\left| \begin{array}{l} a(\beta\ ; u_M, u_M) - \beta^2 c_\infty^2 \displaystyle\int_{\mathbf{R}^2} \varepsilon \mid u_M \mid^2 \ dx = \\ \displaystyle\int_{\mathbf{R}^2} \varepsilon c^2 \{\mid \mathrm{rot} \boldsymbol{u}_M \mid^2 + \mid \nabla u_{3,M} \mid^2\}\ dx + \beta^2 c_\infty^2 \int_{\mathbf{R}^2} \varepsilon \mid u_{3,M} \mid^2 \ dx \\ +\beta^2 \displaystyle\int_{B_R} \varepsilon(c^2 - c_\infty^2) \mid \boldsymbol{u} \mid^2 \ dx. \end{array} \right.$$

Using (5.86) and (5.87) we have

$$\lim_{M \to +\infty} (a(\beta\ ; u_M, u_M) - \beta^2 c_\infty^2 \mid u_M \mid_\varepsilon^2) = \beta^2 \int_{B_R} \varepsilon(c^2 - c_\infty^2) \ \mid \boldsymbol{u} \mid^2 \ dx < 0$$

so that, at least for M large enough, we obtain

$$a(\beta\ ; u_M, u_M) < \beta^2 c_\infty^2 \mid u_M \mid_\varepsilon^2$$

which proves that $s_1(\beta) < c_\infty^2 \beta^2$, $\forall \beta > 0$ and consequently that $\beta_1^* = 0$.

(ib) In order to be complete we have to treat the case where

$$\min_{\boldsymbol{u} \in B(\mathbf{P}_\varepsilon)} \int_{B_R} \varepsilon(c^2 - c_\infty^2) \mid \boldsymbol{u} \mid^2 \ dx = 0. \tag{5.89}$$

Let $\boldsymbol{u} \in B(\mathbf{P}_\varepsilon)$ such that $\int_{B_R} \varepsilon(c^2 - c_\infty^2) \mid \boldsymbol{u} \mid^2 \ dx = 0$. The idea is to take as a test function a small perturbation of the function $\boldsymbol{u}_M$ constructed previously:

$$\boldsymbol{u}_M^\delta = \boldsymbol{u}_M + \delta \boldsymbol{w} \quad \boldsymbol{w} \in H(\mathrm{rot}, \mathrm{div}_\varepsilon\ ; \mathbf{R}^2) \tag{5.90}$$

which leads to define

$$u_M^\delta = \left(\boldsymbol{u}_M^\delta, \frac{1}{\varepsilon\beta} \mathrm{div}(\varepsilon \boldsymbol{u}_M^\delta) \right). \tag{5.91}$$

We are going to prove that $a_\varepsilon(\beta\ ; u_M^\delta, u_M^\delta) < \beta^2 c_\infty^2 \mid u_M^\delta \mid_\varepsilon^2$ if M is sufficiently large, δ sufficiently small and $\boldsymbol{w}$ adequately chosen. In the sequel we shall set $w_3 = \frac{1}{\varepsilon\beta} \mathrm{div}(\varepsilon \boldsymbol{w})$, that we suppose to belong to $H^1(\mathbf{R}^2)$, so that $u_M^\delta = u_M + \delta w$, $w = (\boldsymbol{w}, w_3)$. To estimate $p_\varepsilon(\beta\ ; u_M^\delta, u_M^\delta)$ we simply use the inequality

$$p_\varepsilon(\beta\ ; u_M^\delta, u_M^\delta) \le 2 p_\varepsilon(\beta\ ; u_M, u_M) + 2\delta^2 p_\varepsilon(\beta\ ; w, w). \tag{5.92}$$

Now we compute $c_\varepsilon(\beta\ ; u_M^\delta, u_M^\delta)$. First we have, since $\int_{B_R} \varepsilon(c^2 - c_\infty^2) \mid \boldsymbol{u} \mid^2 \ dx = 0$ and using (5.90),

$$\left| \begin{array}{lcl} \displaystyle\int_{B_R} \varepsilon(c^2 - c_\infty^2) \mid \boldsymbol{u}_M^\delta \mid^2 \ dx & = & 2\delta \displaystyle\int_{B_R} \varepsilon(c^2 - c_\infty^2) \boldsymbol{u} \cdot \boldsymbol{w} \ dx + \delta^2 \int_{B_R} \varepsilon(c^2 - c_\infty^2) \mid \boldsymbol{w} \mid^2 \ dx \\ \displaystyle\int_{B_R} \varepsilon(c^2 - c_\infty^2) \boldsymbol{u}_M^\delta \cdot \nabla u_3^{\delta M} \ dx & = & \delta \displaystyle\int_{B_R} \varepsilon(c^2 - c_\infty^2) \nabla w_3 \cdot \boldsymbol{u} \ dx + \delta^2 \int_{B_R} \varepsilon(c^2 - c_\infty^2) \nabla w_3 \cdot \boldsymbol{w} \ dx. \end{array} \right.$$

Therefore we have

$$\left| \begin{array}{lcl} a(\beta\ ; u_M^\delta, u_M^\delta) - \beta^2 c_\infty^2 \mid u_M^\delta \mid_\varepsilon^2 & \le & 2\delta \displaystyle\int_{B_R} \varepsilon(c^2 - c_\infty^2) \boldsymbol{u} \cdot (\beta^2 \boldsymbol{w} - \beta \nabla w_3) \ dx \\ & + & \delta^2 \displaystyle\int_{B_R} \varepsilon(c^2 - c_\infty^2) \left(\beta^2 \mid \boldsymbol{w} \mid^2 - 2\beta \nabla w_3 \cdot \boldsymbol{w}\right) \ dx \\ & + & 2 p_\varepsilon(\beta\ ; u_M, u_M) + 2\delta^2 p_\varepsilon(\beta\ ; w, w). \end{array} \right. \tag{5.93}$$

We know that $p_\varepsilon(\beta\ ; u_M, u_M) \searrow 0$. Let us choose $\delta_M = (p_\varepsilon(\beta\ ; u_M, u_M))^{\frac{1}{2}} \searrow 0$ (as $\mathbf{P}_\varepsilon \cap L^2(\mathbb{R}^2) = \{0\}$, we have necessarily $p_\varepsilon(\beta\ ; u_M, u_M) > 0$), it is clear that

$$\limsup_{M\to+\infty} \frac{1}{\delta_M}\left\{a(\beta\ ; u_M^{\delta_M}, u_M^{\delta_M}) - \beta^2 c_\infty^2 \mid u_M^{\delta_M} \mid_\varepsilon^2\right\} \leq 2\int_{B_R} \varepsilon(c^2 - c_\infty^2)\boldsymbol{u}\cdot(\beta^2\boldsymbol{w} - \beta\nabla w_3)\, dx\ .$$

To conclude, it suffices to prove that one can construct $\boldsymbol{w}$ such that

$$\int_{B_R} \varepsilon(c^2 - c_\infty^2)\boldsymbol{u}\cdot(\beta^2\boldsymbol{w} - \beta\nabla w_3)\, dx\ <\ 0. \tag{5.94}$$

To prove this, we first note that one can find $\boldsymbol{f}$ in $H(\mathrm{rot}\ ; \mathbb{R}^2)$ such that

$$\int_{B_R} \varepsilon(c^2 - c_\infty^2)\boldsymbol{u}\cdot\boldsymbol{f}\ dx\ <\ 0.$$

Indeed $H(\mathrm{rot}\ ; \mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)^2$ and so $\varepsilon(c^2 - c_\infty^2)\boldsymbol{u}$, which is different from 0, cannot be orthogonal to $H(\mathrm{rot}\ ; \mathbb{R}^2)$. Then let $\boldsymbol{\psi}$ be the unique solution of the problem (apply Lax-Milgram's lemma in the space $H(\mathrm{div}\ ; \mathbb{R}^2)$)

$$\left\{\begin{array}{l} \text{Find } \boldsymbol{\psi} \in H(\mathrm{div}\ ; \mathbb{R}^2)\quad / \\ -\nabla(\frac{1}{\varepsilon}\mathrm{div}\boldsymbol{\psi}) + \beta^2\frac{\boldsymbol{\psi}}{\varepsilon}\ =\ \boldsymbol{f}. \end{array}\right. \tag{5.95}$$

Defining $\boldsymbol{w} = \dfrac{\boldsymbol{\psi}}{\varepsilon}$, we see that $\boldsymbol{w} \in H(\mathrm{rot}\ ; \mathrm{div}\ _\varepsilon ; \mathbb{R}^2)$ (note that $\mathrm{rot}\boldsymbol{w} = \dfrac{1}{\beta^2}\mathrm{rot}\boldsymbol{f}$) and thus setting $w_3 = \dfrac{1}{\varepsilon\beta}\mathrm{div}(\varepsilon\boldsymbol{w})$ we have $w = (\boldsymbol{w}, w_3) \in V_\varepsilon(\beta)$ and $\beta^2\boldsymbol{w} - \beta\nabla w_3 = \boldsymbol{f}$, which concludes the proof of **(ib)**.

(ii) We will not detail the proof which consists in working with two independent functions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in $\mathbf{P}_\varepsilon$ and do exactly the same as in point **(i)** to construct $u_{1,M}$ and $u_{2,M}$ (or $u_{1,M}^\delta$ and $u_{2,M}^\delta$) and then work with a 2-dimensional space of test functions.

A Appendix: Proof of Proposition 5.1 (Equivalence of norms)

We first remark that by definition of R we can find $R_1 < R$ such that the functions $\varepsilon - \varepsilon_\infty$ and $\mu - \mu_\infty$ have their support included in the ball $\{\mid x \mid \leq R_1\}$. Now let ψ be a cut-off function satisfying

$$\left\{\begin{array}{ll} \psi \in C_0^\infty(\mathbb{R}^2), & 0\ \leq\ \psi\ \leq\ 1 \\ \psi = 1 & \text{if } \mid x \mid \leq R_1 \\ \psi = 0 & \text{if } \mid x \mid \geq R\,. \end{array}\right. \tag{A.1}$$

Any function $\boldsymbol{u}$ of $H_\rho(\mathrm{rot}, \mathrm{div}_\varepsilon\ ; \mathbb{R}^2)$ can be decomposed as follows:

$$\left\{\begin{array}{ll} \boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2 & \\ \boldsymbol{u}_1 = \psi\boldsymbol{u} & (\mathrm{supp}\ \boldsymbol{u}_1 \subset B_R) \\ \boldsymbol{u}_2 = (1 - \psi)\boldsymbol{u}. & \end{array}\right. \tag{A.2}$$

First we note that

$$\text{(A.3)} \qquad \int_{\mathbb{R}^2} \rho \mid \boldsymbol{u}_2 \mid^2 \ dx \leq C \int_{\mathbb{R}^2} (\mid \mathrm{rot}\boldsymbol{u}_2 \mid^2 + \mid \mathrm{div}\boldsymbol{u}_2 \mid^2) \ dx,$$

since $\boldsymbol{u}_2$ has its support in the set $\{x \ / \ |x| \geq R_1\}$. Indeed, for any function v in $\mathcal{D}(\mathbb{R}^2 - \overline{B}_{R_1})$, we have the identity

$$\int_{\mathbb{R}^2} (\mid \mathrm{rot}v \mid^2 + \mid \mathrm{div}v \mid^2) \ dx \ = \ \int_{\mathbb{R}^2} \mid \nabla v \mid^2 \ dx \ .$$

Then (A.3) derives from Hardy's inequality (see [23]) and density arguments. As we have the relations

$$\text{(A.4)} \qquad \begin{cases} \mathrm{rot}\boldsymbol{u}_2 = (1-\psi)\mathrm{rot}\boldsymbol{u} - \vec{\mathrm{rot}}\psi \cdot \boldsymbol{u} \\ \\ \mathrm{div}\boldsymbol{u}_2 = (1-\psi)\mathrm{div}\boldsymbol{u} - \nabla\psi \cdot \boldsymbol{u}, \end{cases}$$

we deduce, since supp $\psi \subset \overline{B}_R$ that

$$\text{(A.5)} \quad \left| \begin{array}{l} \displaystyle\int_{\mathbb{R}^2} \mid \mathrm{rot}\boldsymbol{u}_2 \mid^2 \ dx \leq 2 \int_{\mathbb{R}^2} (1-\psi)^2 \mid \mathrm{rot}\boldsymbol{u} \mid^2 \ dx + 2 \parallel \nabla\psi \parallel_\infty^2 \int_{|x| \leq R} \mid \boldsymbol{u} \mid^2 \ dx \\ \displaystyle\int_{\mathbb{R}^2} \mid \mathrm{div}\boldsymbol{u}_2 \mid^2 \ dx \leq 2 \int_{\mathbb{R}^2} \mid 1-\psi \mid^2 \mid \mathrm{div}\boldsymbol{u} \mid^2 \ dx + 2 \parallel \nabla\psi \parallel_\infty^2 \int_{|x| \leq R} \mid \boldsymbol{u} \mid^2 \ dx \end{array} \right.$$

so that we have as $\varepsilon = \varepsilon_\infty$ in supp $(1-\psi)$

$$\text{(A.6)} \quad \int_{\mathbb{R}^2} \rho \mid \boldsymbol{u}_2 \mid^2 \ dx \leq C \left\{ \int_{|x| \leq R} \mid \boldsymbol{u} \mid^2 \ dx + \int_{\mathbb{R}^2} (\mid \mathrm{rot}\boldsymbol{u} \mid^2 + \mid \mathrm{div}(\varepsilon\boldsymbol{u}) \mid^2) \ dx \right\},$$

for an adequate positive constant C (which depends on $\parallel \nabla\psi \parallel_\infty$, $\parallel (1-\psi) \parallel_\infty$ and ε_∞). Now we remark that $\boldsymbol{u}_1$ belongs to the space $H(\mathrm{rot}, \mathrm{div}_\varepsilon \ ; \mathbb{R}^2)$ since it has compact support. We now prove that if

$$\text{(A.7)} \qquad H^R(\mathrm{rot}, \mathrm{div}_\varepsilon \ ; \mathbb{R}^2) \ = \ \{\boldsymbol{v} \in H(\mathrm{rot}, \mathrm{div}_\varepsilon \ ; \mathbb{R}^2) \ , \ \mathrm{supp} \ \boldsymbol{v} \subset B_R\},$$

then there exists a constant $C = C(R)$ such that

$$\text{(A.8)} \ \forall \boldsymbol{v} \in H^R(\mathrm{rot}, \mathrm{div}_\varepsilon \ ; \mathbb{R}^2) \qquad \int_{\mathbb{R}^2} \mid \boldsymbol{v} \mid^2 \ dx \ \leq \ C \int_{\mathbb{R}^2} (\mid \mathrm{rot}\boldsymbol{v} \mid^2 + \mid \mathrm{div}(\varepsilon\boldsymbol{v}) \mid^2) \ dx.$$

Indeed, if not there would exist a sequence $\boldsymbol{v}^n$ whose L^2-norm would be equal to 1 while $\mathrm{rot}\boldsymbol{v}^n$ and $\mathrm{div}(\varepsilon\boldsymbol{v}^n)$ would converge to 0 in L^2. By compactness (cf. Proposition 2.2), we could extract a sequence $\boldsymbol{v}^n$ such that

$$\begin{cases} \boldsymbol{v}^n & \to & \boldsymbol{v} & \text{in} & L^2(B_R)^2 & \text{strongly} \\ \\ \mathrm{div}(\varepsilon\boldsymbol{v}^n) & \to & \mathrm{div}\varepsilon(\boldsymbol{v}) & \text{in} & L^2(\mathbb{R}^2) & \text{strongly} \\ \\ \mathrm{rot}\boldsymbol{v}^n & \to & \mathrm{rot}\boldsymbol{v} & \text{in} & L^2(\mathbb{R}^2) & \text{strongly} \ . \end{cases}$$

The limit function $\boldsymbol{v}$ would satisfy

$$\text{(A.9)} \qquad \begin{cases} \text{supp } \boldsymbol{v} \subset B_R \\ \text{rot}\boldsymbol{v} = \text{div}\varepsilon\boldsymbol{v} = 0 \end{cases}$$

$$\text{(A.10)} \qquad \text{and} \qquad \int_{\mathbf{R}^2} |\, \boldsymbol{v} \,|^2 \; dx = 1.$$

By unique continuation theorem (see section 1.3 (ii)), (A.9) implies that $\boldsymbol{v}$ is identically 0 which contradicts (A.10). Therefore, applying (A.8) to $\boldsymbol{u}_2$ and using the fact that ρ is bounded, we get

$$\text{(A.11)} \qquad \int_{\mathbf{R}^2} \rho \,|\, \boldsymbol{u}_1 \,|^2 \; dx \leq C \int_{\mathbf{R}^2} (|\, \text{rot}\boldsymbol{u}_1 \,|^2 + |\, \text{div}(\varepsilon\boldsymbol{u}_1) \,|^2) \; dx.$$

From the identities

$$\text{(A.12)} \qquad \begin{cases} \text{rot}\boldsymbol{u}_1 = \psi\text{rot}\boldsymbol{u} + \vec{\text{rot}}\psi \cdot \boldsymbol{u} \\ \text{div}(\varepsilon\boldsymbol{u}_1) = \psi\text{div}(\varepsilon\boldsymbol{u}) + \nabla\psi \cdot \varepsilon\boldsymbol{u} \end{cases}$$

and the fact that $\nabla\psi$ has its support in B_R we deduce

$$\text{(A.13)} \quad \int_{\mathbf{R}^2} \rho \,|\, \boldsymbol{u}_1 \,|^2 \; dx \leq C \left\{ \int_{|x| \leq R} |\, \boldsymbol{u} \,|^2 \; dx + \int_{\mathbf{R}^2} (|\, \text{rot}\boldsymbol{u} \,|^2 + |\, \text{div}(\varepsilon\boldsymbol{u}) \,|^2) \; dx \right\}.$$

Finally, Proposition 5.1 follows from (A.3) and (A.13) since we have $\int_{\mathbf{R}^2} \rho \,|\, \boldsymbol{u} \,|^2 \; dx \leq 2\left(\int_{\mathbf{R}^2} \rho \,|\, \boldsymbol{u}_1 \,|^2 \; dx + \int_{\mathbf{R}^2} \rho \,|\, \boldsymbol{u}_2 \,|^2 \; dx \right)$.

B Appendix: Proof of Lemma 5.2 (Density of $V_{\varepsilon,c}(\beta)$ in $V_{\rho,\varepsilon}(\beta)$)

We use the same truncation functions as these in the thesis of J. Giroire [16] (see also P. Bolley and J. Camus [14]). They are defined for $n \geq 1$, by

$$\text{(B.1)} \qquad \psi^n(x) = \begin{cases} \psi(\dfrac{n}{\log |x|}) & \text{if } |x| > 1, \\ 1 & \text{if } |x| \leq 1 \end{cases}$$

where ψ belongs to $C^\infty([0, +\infty[)$ and satisfies

$$\text{(B.2)} \qquad \begin{cases} \psi(t) = 0 & \text{if } t \in [0, 1] \\ 0 \leq \psi(t) \leq 1 & \text{if } t \in [1, 2] \\ \psi(t) = 1 & \text{if } t \in [2, +\infty[. \end{cases}$$

So $\psi^n \in C^\infty(\mathbf{R}^2)$ and satisfies

$$\text{(B.3)} \qquad \begin{cases} \text{supp } \psi^n \subset B(0, e^n), & 0 \leq \psi^n(x) \leq 1 \\ \psi^n(x) = 1 & \text{if } |x| \leq e^{\frac{n}{2}} \end{cases}$$

The most important property of ψ^n lies in the following estimate valid for any multi-index $\alpha \in \mathbb{N}^* \times \mathbb{N}^*$ and any $n \geq 2$ (see [16]):

$$\text{(B.4)} \qquad |\mathcal{D}^\alpha \psi^n(x)| \leq C_\alpha \rho^{\frac{1}{2}}(x) \quad \text{for } x \text{ such that } e^{\frac{n}{2}} \leq |x| \leq e^n,$$

where the constant C_α does not depend on n and where ρ is the weight function defined in (5.12). Let $u = (\boldsymbol{u}, u_3)$ be in $V_{\rho,\varepsilon}(\beta)$ and set

$$\boldsymbol{u}^n = \psi^n \boldsymbol{u} \quad \text{and} \quad u_3^n = \frac{1}{\varepsilon\beta} \text{div}(\varepsilon \boldsymbol{u}^n).$$

By construction $u^n = (\boldsymbol{u}^n, u_3^n)$ has a compact support and for n large enough,

$$u^n \in V_{\varepsilon,c}(\beta).$$

We show that (u^n) converges to u in $V_{\rho,\varepsilon}(\beta)$. For this we must prove that $\boldsymbol{u}^n$ tends to $\boldsymbol{u}$ in $H_\rho(\text{rot}, \mathbf{R}^2)$ and that u_3^n tends to u_3 in $H^1(\mathbf{R}^2)$. Let us first consider $\boldsymbol{u}^n$, by Lebesgue's theorem we clearly have (since $\psi^n(x) \in [0,1]$)

$$\text{(B.5)} \qquad \boldsymbol{u}^n \longrightarrow \boldsymbol{u} \quad \text{in} \quad L^2(\mathbf{R}^2, \rho dx)^2.$$

Then

$$\text{rot}\boldsymbol{u}^n = \text{rot}\boldsymbol{u}\psi^n - \vec{\text{rot}}\psi^n \cdot \boldsymbol{u}.$$

Now, still by Lebesgue's theorem,we have

$$\text{(B.6)} \qquad \psi^n \text{rot}\boldsymbol{u} \longrightarrow \textit{rot}\boldsymbol{u} \quad \text{in} \quad L^2(\mathbf{R}^2).$$

We thus have to show that $\|\vec{\text{rot}}\psi^n \cdot \boldsymbol{u}\|_{L^2(\mathbf{R}^2)} \to 0$. We have

$$\|\vec{\text{rot}}\psi^n \cdot \boldsymbol{u}\|^2_{L^2(\mathbf{R}^2)} \leq \int_{e^{\frac{n}{2}} \leq |x| \leq e^n} |\nabla\psi^n|^2 |\boldsymbol{u}|^2 dx.$$

Using (B.4)

$$\|\vec{\text{rot}}\psi^n \cdot \boldsymbol{u}\|^2_{L^2(\mathbf{R}^2)} \leq C \int_{e^{\frac{n}{2}} \leq |x| \leq e^n} \rho |\boldsymbol{u}|^2 dx,$$

and $\rho^{\frac{1}{2}}\boldsymbol{u}$ belonging to $L^2(\mathbf{R}^2)^2$, we have

$$\text{(B.7)} \qquad \vec{\text{rot}}\psi^n \cdot \boldsymbol{u} \longrightarrow 0 \quad \text{in} \quad L^2(\mathbf{R}^2).$$

We deduce from (B.6) and (B.7) that

$$\text{(B.8)} \qquad \text{rot}\boldsymbol{u}^n \longrightarrow \text{rot}\boldsymbol{u} \quad \text{in} \quad L^2(\mathbf{R}^2).$$

Concerning u_3^n we note that

$$u_3^n = \psi^n u_3 + \frac{1}{\beta} \nabla\psi^n \cdot \boldsymbol{u}.$$

Reasoning as before we have

$$\nabla\psi^n \cdot \boldsymbol{u} \longrightarrow 0 \quad \text{in} \quad L^2(\mathbf{R}^2),$$

and by Lebesgue's theorem

(B.9)
$$\psi^n u_3 \longrightarrow u_3 \quad \text{in} \quad L^2(\mathbf{R}^2).$$

so that

(B.10)
$$u_3^n \longrightarrow u_3 \quad \text{in} \quad L^2(\mathbf{R}^2).$$

Moreover we have

$$\frac{\partial u_3^n}{\partial x_i} = \psi^n \frac{\partial u_3}{\partial x_i} + \frac{\partial \psi^n}{\partial x_i} u_3 + \frac{1}{\beta}\left(\frac{\partial}{\partial x_i}(\nabla\psi^n) \cdot \boldsymbol{u} + \nabla\psi^n \cdot \frac{\partial \boldsymbol{u}}{\partial x_i}\right).$$

Once again by Lebesgue's theorem, we obtain

(B.11)
$$\psi^n \frac{\partial u_3}{\partial x_i} \longrightarrow \frac{\partial u_3}{\partial x_i} \quad \text{in} \quad L^2(\mathbf{R}^2).$$

As for $\vec{\mathrm{rot}}\psi^n \cdot \boldsymbol{u}$ in (B.7), we prove that

(B.12)
$$\frac{\partial \psi^n}{\partial x_i} u_3 \longrightarrow 0 \quad \text{in} \quad L^2(\mathbf{R}^2).$$

As long as $e^{\frac{n}{2}} > R$, $\dfrac{\partial u}{\partial x_i}$ belongs to $L^2(|x| > R)^2$ and supp $\nabla\psi^n \subset \{|x| > R\}$, which allows us to prove that

(B.13)
$$\nabla\psi^n \cdot \frac{\partial \boldsymbol{u}}{\partial x_i} \longrightarrow 0 \quad \text{in} \quad L^2(\mathbf{R}^2).$$

It remains to look at the term $\dfrac{\partial}{\partial x_i}(\nabla\psi^n) \cdot \boldsymbol{u}$. But according to (B.4), we have

(B.14)
$$\left\{\begin{array}{rcl} \|\dfrac{\partial}{\partial x_i}(\nabla\psi^n) \cdot \boldsymbol{u}\|^2_{L^2(\mathbf{R}^2)} & \leq & \displaystyle\int_{e^{\frac{n}{2}} \leq |x| \leq e^n} \left(|\frac{\partial^2}{\partial x_i \partial x_1}\psi^n|^2 + |\frac{\partial^2}{\partial x_i \partial x_2}\psi^n|^2\right) |\boldsymbol{u}|^2 \, dx \\ & \leq & C \displaystyle\int_{e^{\frac{n}{2}} \leq |x| \leq e^n} \rho |\boldsymbol{u}|^2 \, dx. \end{array}\right.$$

The right hand side member tends to 0 since $\rho^{\frac{1}{2}}\boldsymbol{u} \in L^2(\mathbf{R}^2)^2$. This concludes the proof of Lemma 5.2.

Remark B.1
- *We can use the same process to prove that the compactly supported functions of $H(rot, div_\epsilon, \mathbf{R}^2)$ are dense in $H_\rho(rot, div_\epsilon, \mathbf{R}^2)$.*
- *Note that the choice of the truncation functions ψ^n is original and allow us to have the estimations (B.4), which we cannot obtain if we use classical truncation functions like $\psi^n(x) = \psi(\dfrac{|x|}{n})$.*

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